Lab Class ML:VI

By 2018-01-17 solutions for the following exercises have to be submitted: 1, 3.

Exercise 1 : Perceptron Learning

Solve the following problems on learning boolean functions with perceptrons. Use the values 0 for *false* and 1 for *true*, and the threshold function $\varphi(x) = \mathsf{max}(sign(x), 0)$.

(a) Design a single perceptron with two inputs x_A and x_B. This perceptron shall implement the boolean formula $A \wedge \neg B$ with a suitable function $y(x_A, x_B)$. Hint: to start, determine the training data and draw it, and a suitable decision boundary, in a coordinate system; then, determine a set of suitable weights $\mathbf{w} = (w_0, w_1, w_2)$.

(b) Train the perceptron from (a) with two iterations of the batch gradient descent algorithm, with a learning rate η of 0.1 and the weights initialized with $w_0 = -0.5$ and $w_1 = w_2 = 0.5$. Use the following examples in the given order:

x_1	x_2	$c(\mathbf{x})$
0	0	0
0	1	0
1	0	1
1	1	0

(c) Why can the boolean formula A *XOR* B not be learned by a single perceptron? Justify your answer with a drawing.

Exercise 2 : Gradient Descent

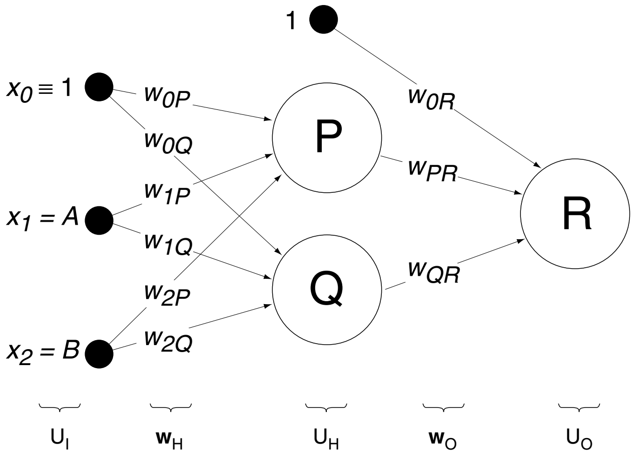
(a) What are the differences between the perceptron training rule and the gradient descent method?

(b) What are the requirements for gradient descent being successful as a learning algorithm?

(c) What are the differences between the batch and the incremental (stochastic) gradient descent?

Exercise 3 : $\boxed{\text{P}}$ Multilayer Neural Networks

Your task is to approximate the boolean formula A *XOR* B using a two-layer neural network with the following architecture:



Following the notation used in the lecture:

$$\begin{aligned}
U_I &= \{x_0, x_1, x_2\} \\
U_H &= \{x_0, y_P, y_Q\} \\
U_O &= \{y_R\} \\
\mathbf{w} &= \mathbf{w}_H \cup \mathbf{w}_O \\
\mathbf{w}_H &= \{w_{0P}, w_{0Q}, w_{1P}, w_{1Q}, w_{2P}, w_{2Q}\} \\
\mathbf{w}_O &= \{w_{0R}, w_{PR}, w_{QR}\}
\end{aligned}$$

For thresholding, we use the sigmoid function $\sigma(x) = \frac{1}{1+e^{-x}}$. Hence, for a given input x and weight vector $\mathbf{w}$, the network output y_R can be written as:

$$\begin{aligned}
y_R(\mathbf{x}, \mathbf{w}) &= \sigma\left(w_{0R} + w_{PR} \cdot y_P(\mathbf{x}, \mathbf{w}_H) + w_{QR} \cdot y_Q(\mathbf{x}, \mathbf{w}_H)\right) \\
&= \sigma\left(w_{0R} + w_{PR} \cdot \sigma\left(w_{0P} + w_{1P}x_1 + w_{2P}x_2\right) + w_{QR} \cdot \sigma\left(w_{0Q} + w_{1Q}x_1 + w_{2Q}x_2\right)\right)
\end{aligned}$$

(a) For the 9 elements of $\mathbf{w}$, first determine a set of suitable values by hand, so that all examples are classified correctly. Assume the classification rule

$$\hat{c}(x) = \begin{cases} 0 & \text{if} \quad y_R(\mathbf{x}, \mathbf{w}) \leq 0.5 \\ 1 & \text{if} \quad y_R(\mathbf{x}, \mathbf{w}) > 0.5 \end{cases}$$

Hints: first determine the training set $D = \{(x, c(x)) \mid x = (1, A, B); c(x) = A \text{ XOR } B)\}$ for all possible A, B. Then, decompose the XOR function into simpler boolean functions, and set the weights $\mathbf{w}_H$ so that y_P and y_Q operate accordingly. Finally, set the weigths $\mathbf{w}_O$ to get the correct y_R.

(b) Implement the weight adaptation via batch gradient descent to find a set of weights for the XOR problem automatically. Employ the error function

$$\mathit{Err}(\mathbf{w}) = \frac{1}{2} \sum_{(\mathbf{x}, \mathbf{c}(\mathbf{x})) \in D} (c(\mathbf{x}) - y_R(\mathbf{x}))^2$$

Use the pseudocode for backpropagation with incremental gradient descent given in the Lecturenotes for guidance, and consider the following hints:

- Represent the training set as a pair of two-dimensional numpy arrays with shapes $(4, 3)$ and $(4, 1)$, e.g.:

```
inputs = array([[1, 0, 0],                 outputs = array([[0],
                [1, 0, 1],                                  [1],
                  ...    ])                                 ...])
```

- Represent the two weight vectors $\mathbf{w}_H$ and $\mathbf{w}_O$ as two-dimensional numpy arrays `W_H` and `W_O` with shapes $(3, 2)$ and $(3, 1)$, and initialize them with random values in the range $[-0.5, 0.5]$ (use an appropriate function from `numpy.random`). Define the `sigmoid` function using numpy array operations.
- Compute the forward pass as in the equation for y_R given above. Convince yourself that the hidden layer activations $y_H = (y_P, y_Q)$ can be computed for the entire training set with a single dot product:

  ```
  y_H = sigmoid( np.dot(inputs, W_H) )
  ```

  However, note that y_H must receive an extra column of ones before y_R can be computed in a similar fashion.
- The goal of the backpropagation pass then is to compute an adjustment to each element of $\mathbf{w}$ according to $\frac{\partial Err}{\partial \mathbf{w}}$. Refer to the lecture slides, but keep the difference between incremental and batch gradient descent in mind.
- Run the batch gradient descent for 10000 iterations for different random starting weights, and observe how *Err* changes over time. If your implementation fails to converge to a correct solution a lot of the time, adjust the learning rate η, and consider using the *momentum* weight adaptation discussed in the lecture.