Fundamentals of Imaging
Lenses

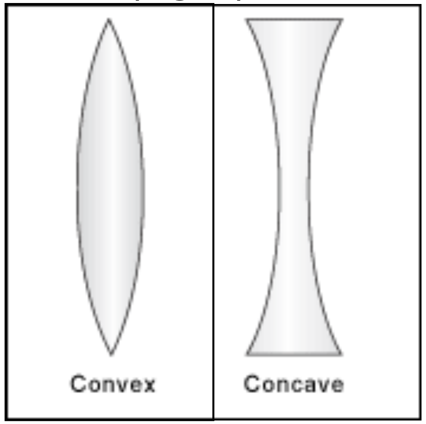
Prof. Dr. Charles A. Wüthrich,
Fakultät Medien, Medieninformatik
Bauhaus-Universität Weimar
caw AT medien.uni-weimar.de

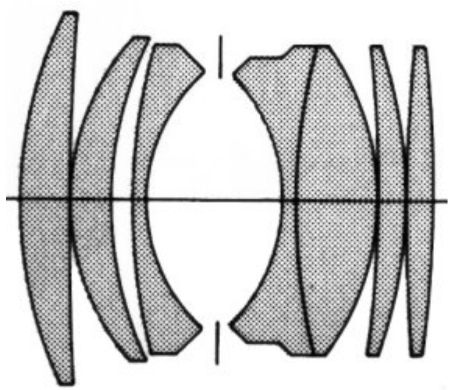
This slide pack

- In this part, we will introduce lenses

Lenses

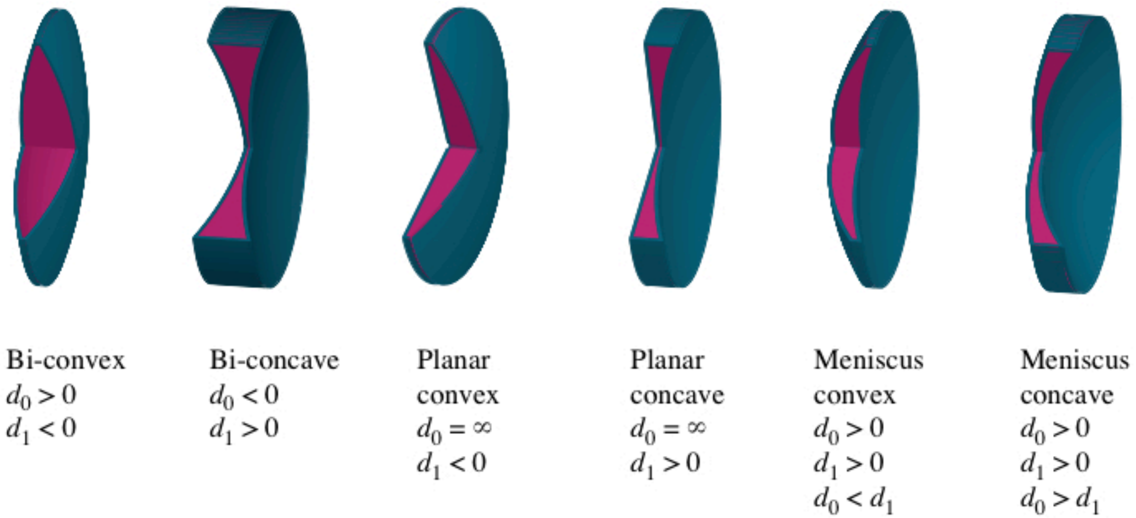
- The components of an optical system consist of
  - aperture ring
  - Refractive elements
- Lenses:
  - Simple (left): single element
    Characteristics:
    - Refraction index
    - Shape of front+back
    - Often coated to improve optical properties
  - Compound (right): multiple lenses





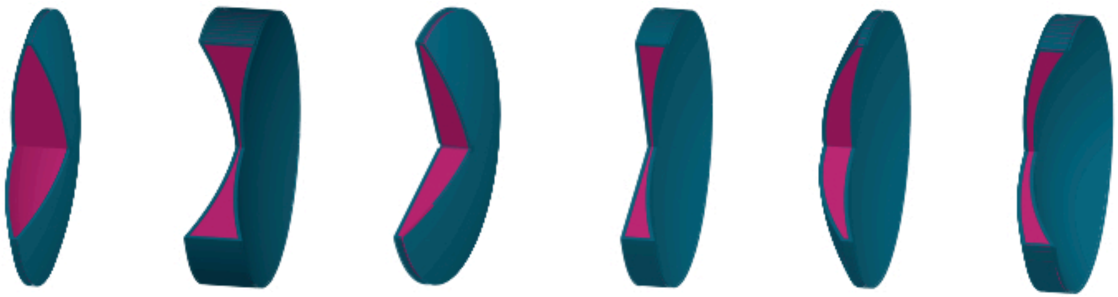
Lenses

- Surface shapes:
  - Planar
  - Spherical
  - Aspherical: some surface which is not a sphere
- Call
  - d_0: radius surface facing object plane
  - d_1: radius surface facing image plane
- Depending on positive or negative radius, one can have different single lens types



Lenses

- Convex lenses direct light towards the optical axis: convergent or positive
- Concave lenses do the opposite and are called divergent or negative



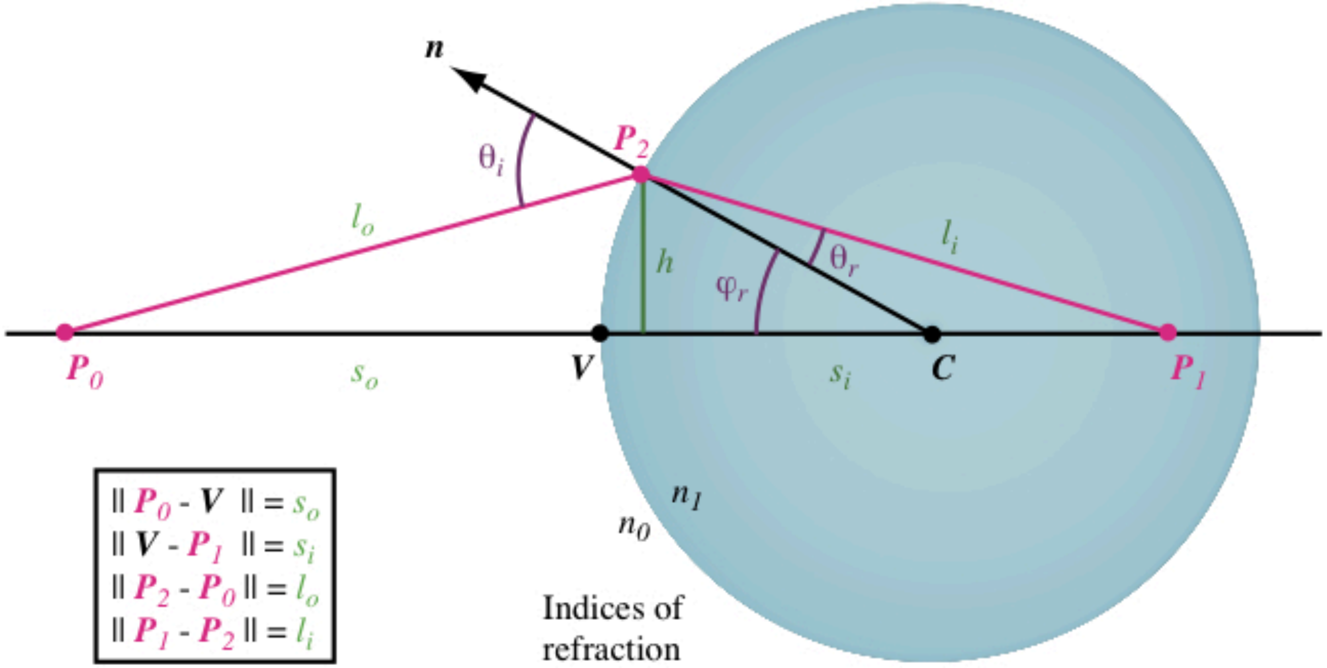
Spherical surface

- Fermat's principle ⇒ optical path length of ray is $n_0 l_o + n_1 l_i$
- Path length of l_o and l_i is

$$l_o = \sqrt{d^2 + (s_o + d)^2 - 2d\,(s_o + d)\cos(\phi)},$$
$$l_i = \sqrt{d^2 + (s_i - d)^2 + 2d\,(s_i - d)\cos(\phi)}.$$

- Substituting in the 1st

$$n_0 \sqrt{d^2 + (s_o + d)^2 - 2d\,(s_o + d)\cos(\phi)}$$
$$+ n_1 \sqrt{d^2 + (s_i - d)^2 + 2d\,(s_i - d)\cos(\phi)}.$$



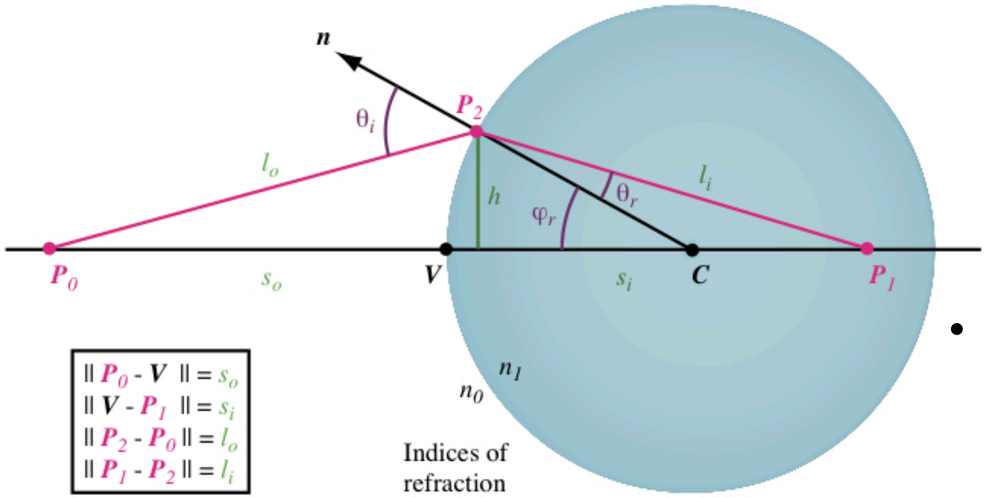
Spherical surface

- Applying Fermat's principle by using Φ as the position variable:

$$\frac{n_0 d\,(s_o + d)\sin(\phi)}{2l_o} - \frac{n_1 d\,(s_i - d)\sin(\phi)}{2l_i} = 0.$$

thus: $\frac{n_0}{l_o} + \frac{n_1}{l_i} = \frac{1}{d}\left(\frac{n_1 s_i}{l_i} - \frac{n_0 s_o}{l_o}\right)$

- Rays from P_0 to P_1 with one refraction obey this law



- Remember, if the angle is too flat, refraction turns into reflection
- Under the hypotheses of Gaussian optics $\cos(\Phi)\approx 1$.
- If we consider only paraxial rays

$$l_o \approx s_o,$$
$$l_i \approx s_i.$$

thus the eq. on the left becomes

$$\frac{n_0}{s_o} + \frac{n_1}{s_i} = \frac{n_1 - n_0}{d}$$

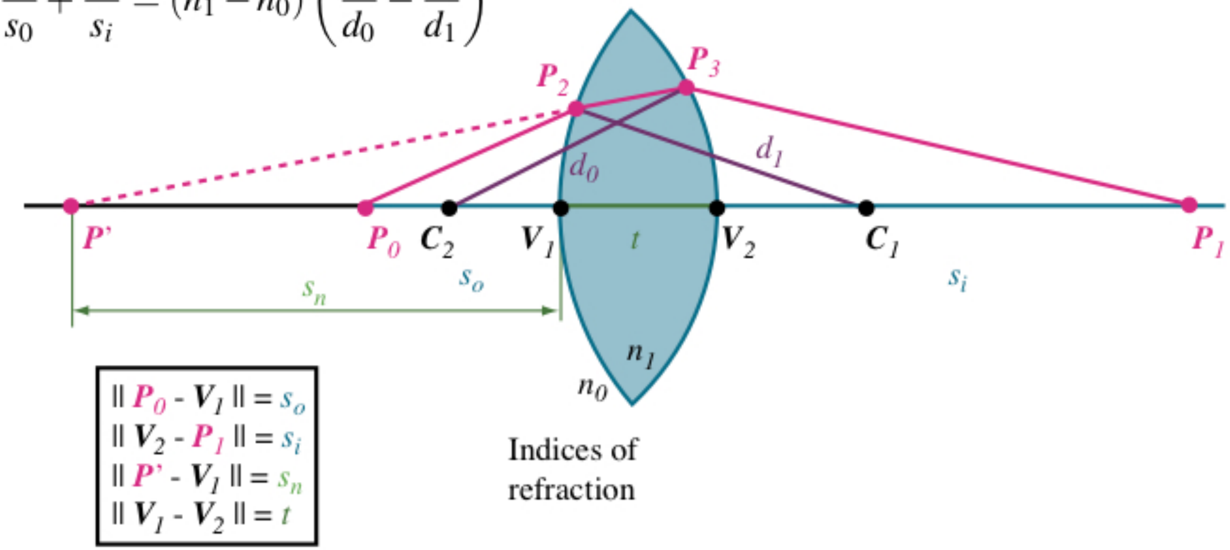
and if the image point is at ∞, I.e. if $s_i = \infty$, then:

  – Object focal length: $f = s_o = \frac{n_0}{d\,(n_1 - n_0)}$

- Similarly, image focal length is obtained for $s_0 = \infty$.
  – Image focal length: $f' = s_i = \frac{d\,n_1}{n_1 - n_0}$

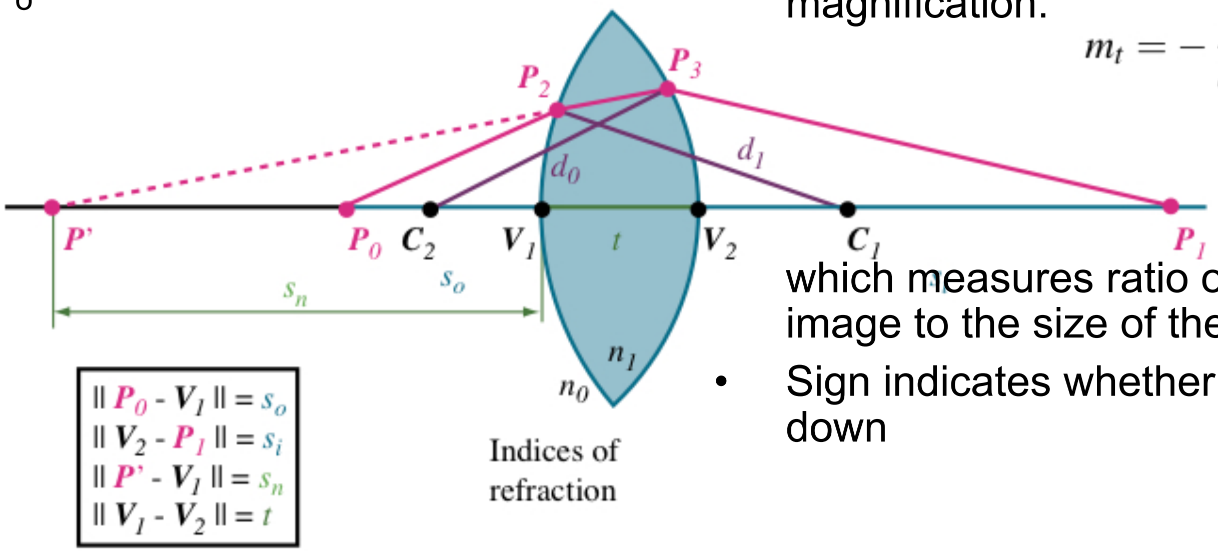
Thin lenses

- However, lenses have a front and back surface
  - Spherical surfaces of radius d_0 and d_1.
  - Analyzing front+back we have
    $$\frac{n_0}{s_0}+\frac{n_0}{s_i}=(n_1-n_0)\left(\frac{1}{d_0}-\frac{1}{d_1}\right)+\frac{n_1 t}{(s_n-t)\,s_n}$$
  - If the lens is thin, then t≈0 ⇒
    $$\frac{n_0}{s_0}+\frac{n_0}{s_i}=(n_1-n_0)\left(\frac{1}{d_0}-\frac{1}{d_1}\right)$$
- If lens is surrounded by air, then $n_0 \approx 1$ ⇒ *lens maker's formula*
  $$\frac{1}{s_o}+\frac{1}{s_i}=(n_1-1)\left(\frac{1}{d_0}-\frac{1}{d_1}\right)$$



Thin lenses

- If object distance $s_o=\infty$, the image distance becomes the image focal length $s_i=f_i$.
- Conversely, for points projected on an infinitely far away image plane, object focal length becomes $s_o=f_o$.
- But lens is thin, so we can set $f_i=f_o$ and call it f



- So, we have $\frac{1}{f} = (n_1 - 1)\left(\frac{1}{d_0} - \frac{1}{d_1}\right)$
- Note: all rays at distance f in front of the lens, and passing through focal point, will be parallel after the lens (*collimated light*)
- Combining we find the *Gaussian lens formula*: $$\frac{1}{f} = \frac{1}{s_o} + \frac{1}{s_i}$$
- Related to this: transverse (lateral) magnification: $$m_t = -\frac{s_i}{s_o}$$ which measures ratio of size of the image to the size of the object
- Sign indicates whether it is upside down

Thick lenses

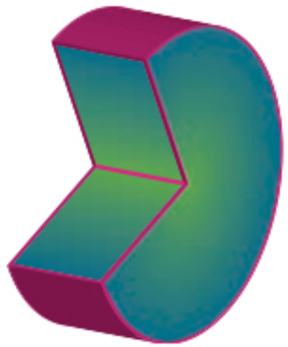
- Most lenses are not thin, they are thick
- We can think they are two spherical surfaces at a distance *t*
- Such lenses behave like the optical systems seen before, with 6 cardinal points
- If focal length is measured WRT principal planes, then

$$\frac{1}{f} = (n_1 - 1)\left(\frac{1}{d_0} - \frac{1}{d_1} + \frac{(n_1 - 1)t}{n_1\, d_0\, d_1}\right)$$

Gradient lenses

- Lens design characteristics considered up to now:
  - Dielectric material
  - Curvature of lens
- Placing more elements behind each other give additional flexibility
- However, one could build lenses having different refraction indexes at different places
- These are called *gradient index lenses*
- Obtained through immersing into salt solutions, which ionize and change refraction index

- In this case, one can use cylinders as lenses: these are called GRIN lenses
- Much more difficult to evaluate optically
- Raytracing may be used for this evaluation



Lens aberration

- We did simple lens approximations, 1st order
- Lens computations must be higher order
- Often, ray tracing is used to evaluate lens design, which works well in theory
- However, deviations from ideal conditions occur: they are called *aberrations*
- These are of two types
  - Chromatic aberrations: the index of refraction is wavelength dependent: *purple fringing*
  - Rainbow-colour inaccuracies on edges

Lens aberration

Curtesy Chem Kurmuk, pentaxforums.com

Lens aberration

- Or one can have monochromatic aberrations
- Out of focus:
  - Spherical aberration
  - Astigmatism
  - Coma
- Warped image
  - Distortion
  - Petzval field distortion
- Reason for these aberrations:
  - We approximated sinus and cosinus linearly
    - sin=linear
    - cos=constant
  - Assumed paraxial rays

- This is a pretty rough approximation: we cut Taylor series to first term:

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

- One could use higher order terms:
  - if one uses 2nd term, one obtains so-called *Third order theory*
  - Nicer, but higher complexity
- Aberrations here: due to Gaussian approximation

Spherical aberrations

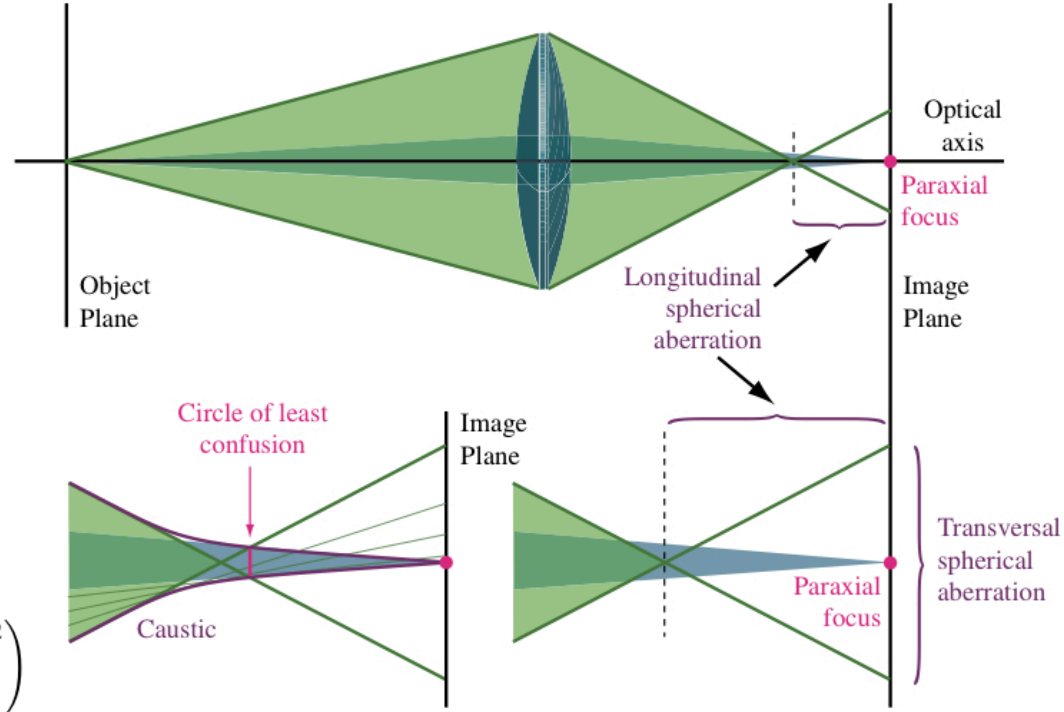
- For spherical lenses, we assumed that the ray has same length as path from object to image plane (on optical axis)
- If we keep term of 2nd degree, then equation

$$\frac{n_0}{s_o} + \frac{n_1}{s_i} = \frac{n_1 - n_0}{d}$$

becomes

$$\frac{n_0}{s_o} + \frac{n_1}{s_i} = \frac{n_1 - n_0}{d} + h^2\left(\frac{n_0}{2s_o}\left(\frac{1}{s_0} + \frac{1}{d}\right)^2 + \frac{n_1}{2s_i}\left(\frac{1}{d} - \frac{1}{s_i}\right)^2\right)$$

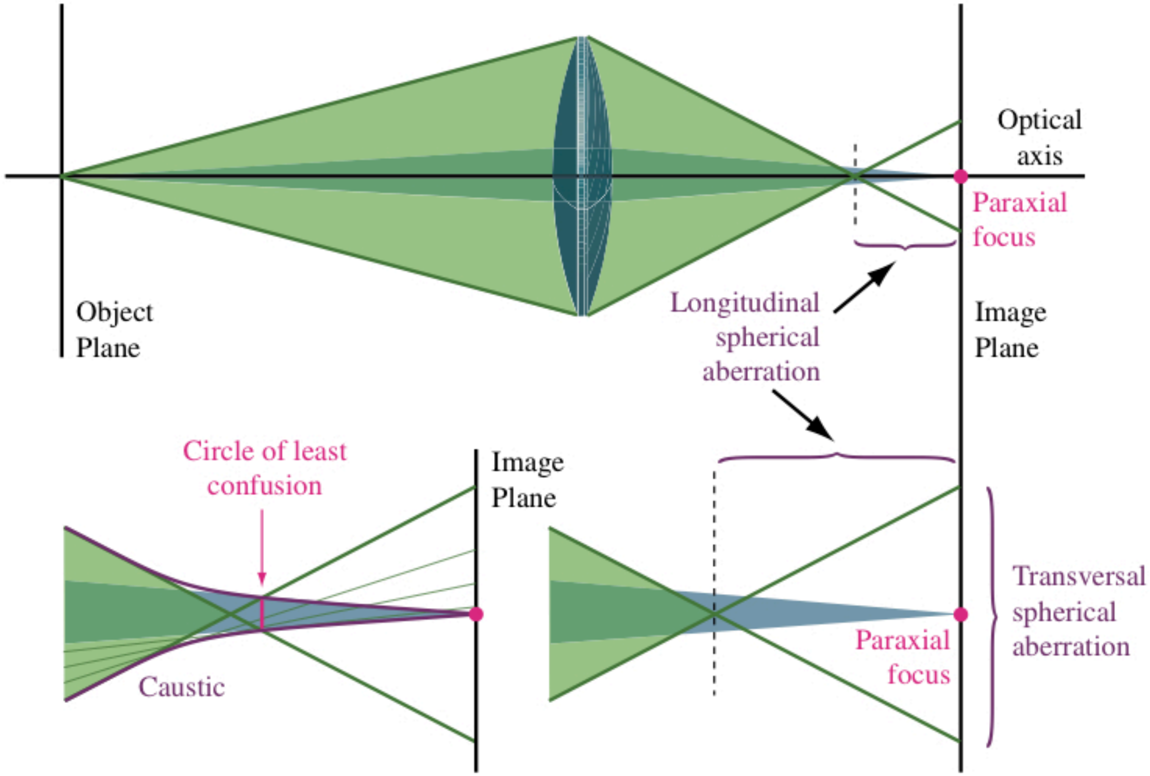
the extra term depends on h^2 with
  - h=distance from point of lens where ray meets optical axis



- Light at lens border is focused nearer
- Defined as *Spherical aberration*

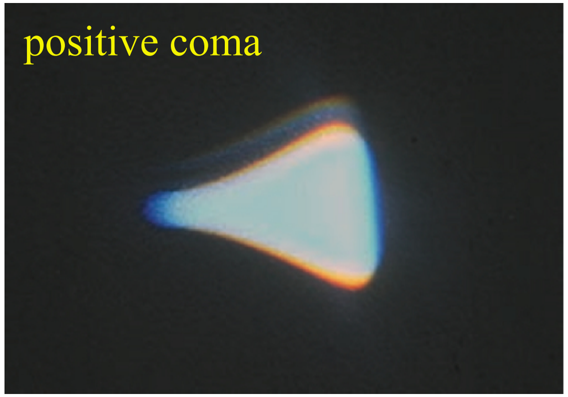
Spherical aberrations

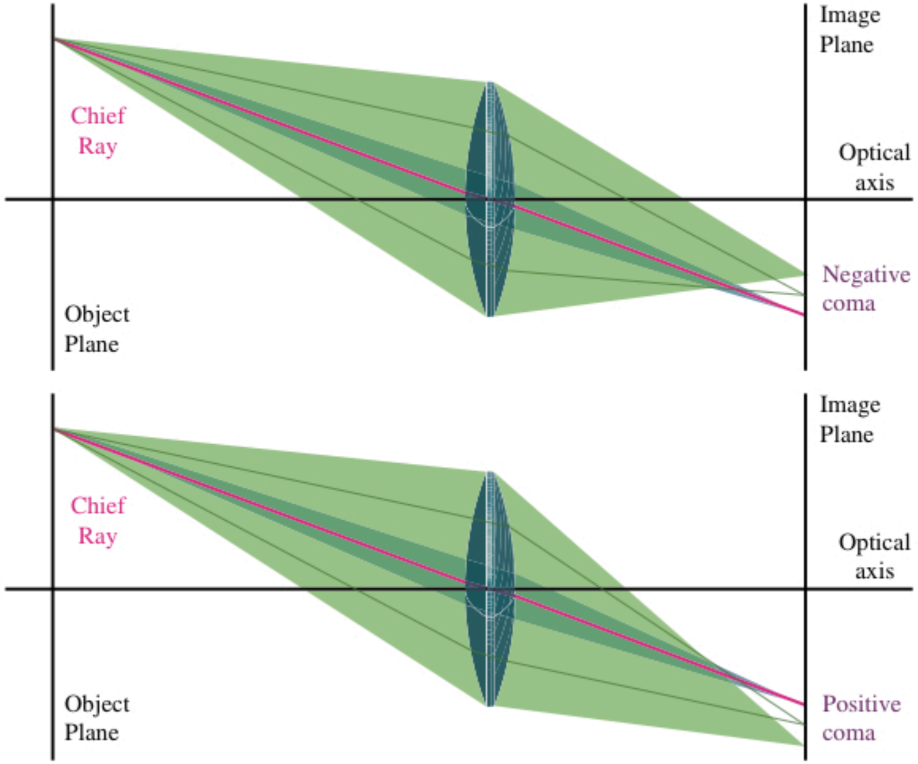
- Rays intersect optical axis over a length, not a point: longitudinal aberration.
- The rays will intersect image plane on a region: transversal aberration.
- With spherical aberration, rays form curved convex hull: *caustic*
- Diameter of projected spot smallest at circle of least confusion



Coma

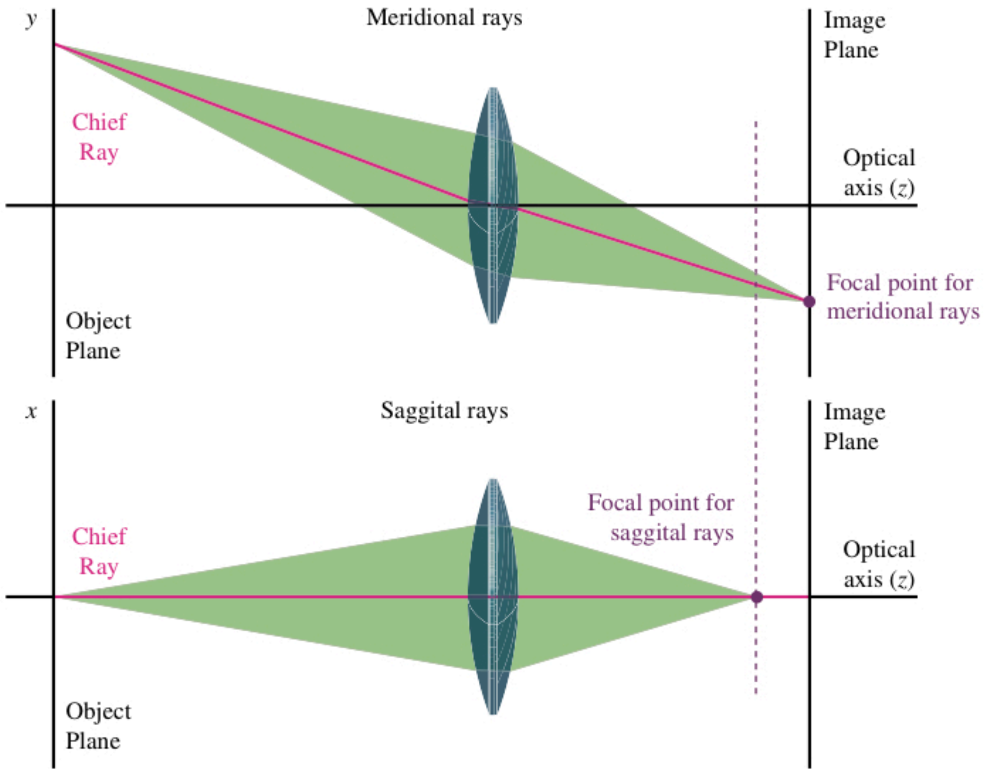
- Principal planes are well approximated only near the optical axis
- Further away, they are curved
- The effect is called *coma*:
  - Marginal ray focus farther than principal ray: *positive coma*
  - If closer, *negative coma*

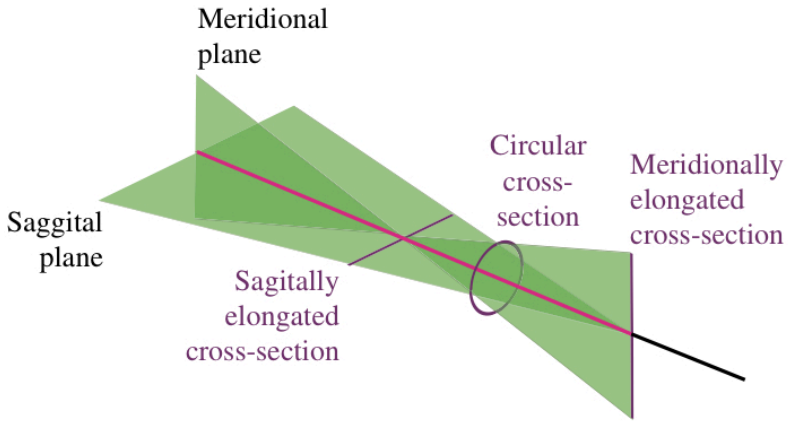




Astigmatic aberration

- Occurs for off-axis object points
- Meridional plane was defined by object point and optical axis
- Chief ray lies in this plane but refracts at lens borders
- Sagittal plane:
  - ⊥ meridional plane
  - Made by set of planar segments, which intersect the chief ray

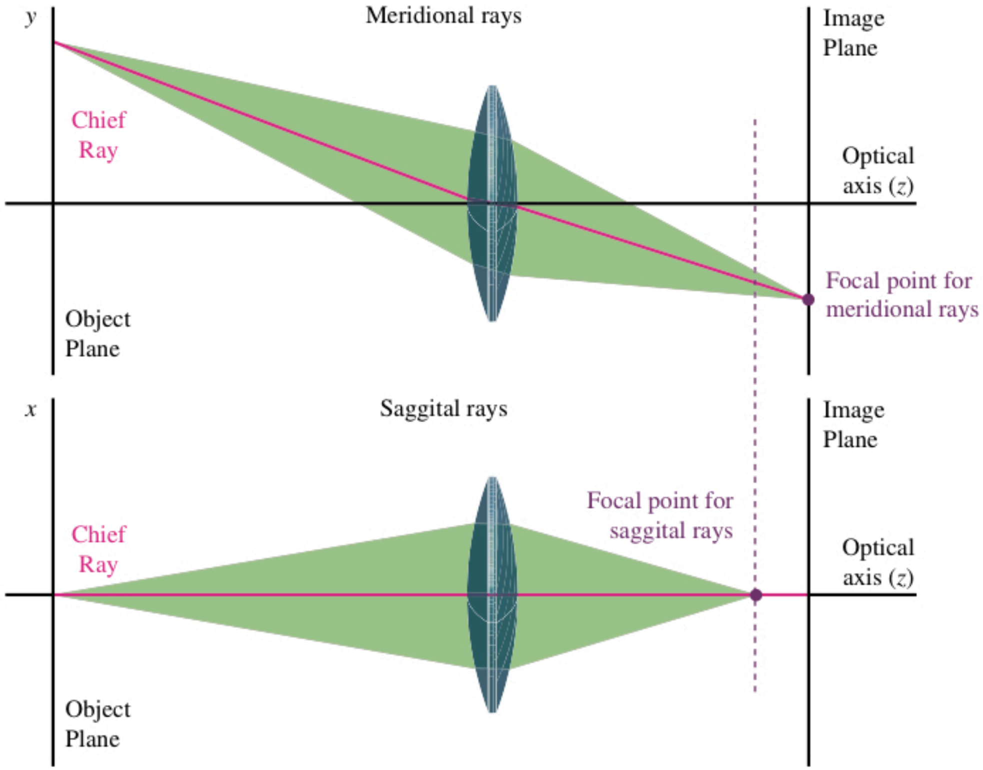


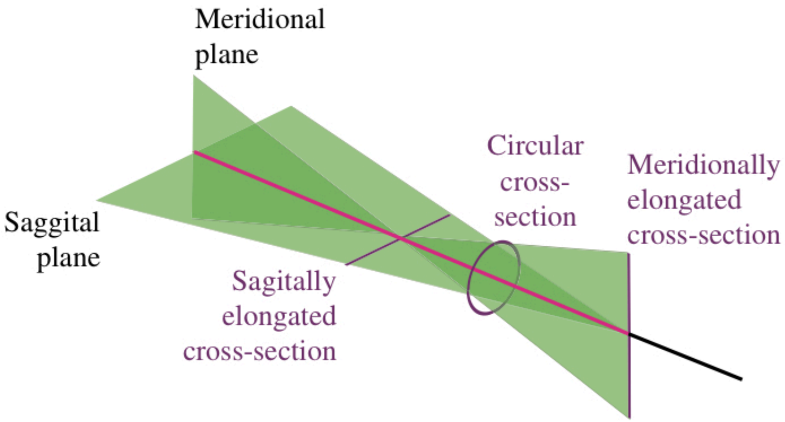


- Consider:
  - ray bundle in merid. Plane
  - ray bundle in sagittal plane
  - Path length could be different

Astigmatic aberration

- At sagittal focal point meridional rays will not have converged :
  - elongated focal point,
  - Elongated meridional focal point.
- For rays neither sagittal nor meridional, focal point will be in between the sagittal and meridional focal points.
- Somewhere between two focal points the cross-section of rays is circular





- When we have astigmatism, this circle is the place of sharpest focal point: *circle of least confusion*

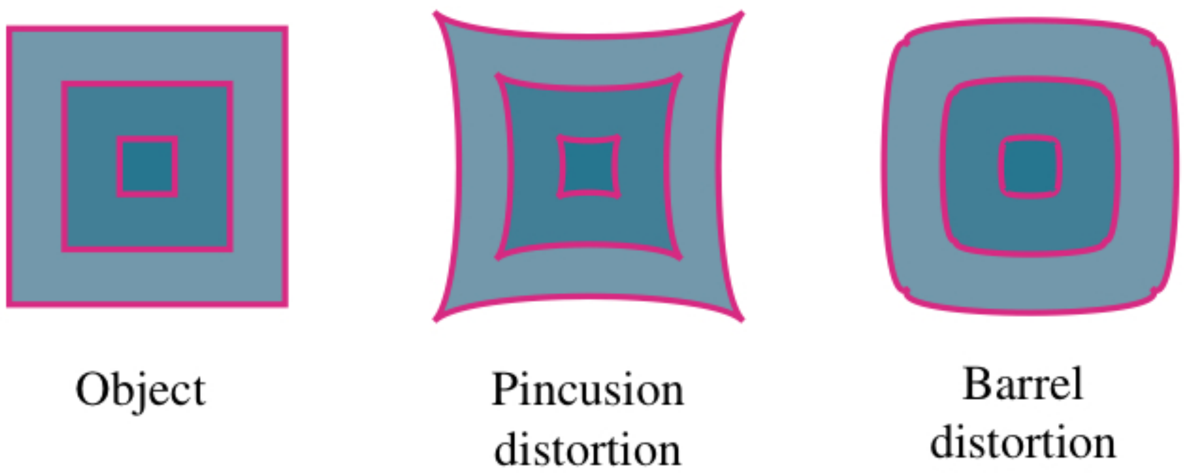
Petzval field curvature

- For spherical lenses, object and image planes are not planar, but:
  - Positive lens: curve inwards
  - Negative lens: curve outwards
  - *Petzval field curvature*
- If a flat image plane is used, I.e. on a sensor, the image will only be sharp on optical axis
- One can correct this by combining positive and negative lenses

- Example: correct inward curvature of positive lens with negative length near focal point of positive lens: *field flattener*.

Distortion

- Distortion is due to lateral magnification of lens:
  - Lateral magnification is not constant as assumed before
- Pincushion distortion:
  lateral magnification increases with distance to optical axis
  - Usually positive lenses generate it
- Barrel distortion: lat. mag. decreases with distance to optical axis
  - Usually negative lenses generate it



Cromatic aberrations

- Materials have wavelength-dependent refraction index, which influences focal length
  - Thus focal length a white light beam lies closer for blue rays than to red light
  - Distance between these two points on optical axis is the *axial chromatic aberration*
  - These rays will hit image at different position (*lateral chromatic aberration*)
  - Can be corrected by using thin lenses with different refractive indexes (*thin achromatic doublets*):
    - If d is the distance between the lenses, wavelength dep. focal length is $f_1(\lambda)$ and $f_2(\lambda)$, refraction indexes $n_1(\lambda)$ and $n_2(\lambda)$ wavelength dependent focal length $f(\lambda)$ is then given by

$$\frac{1}{f(\lambda)} = \frac{1}{f_1(\lambda)} + \frac{1}{f_2(\lambda)} - \frac{d}{f_1(\lambda)\, f_2(\lambda)}$$

Cromatic aberrations

- If the index of refraction of surrounding medium is 1, then

$$\frac{1}{f_1(\lambda)} = k_1\,(n_1(\lambda) - 1)$$

$$\frac{1}{f_2(\lambda)} = k_2\,(n_2(\lambda) - 1)$$

is wavelength dependent focal length. Here, we replaced factor depending on front and back radius with constants k_1,k_2.

- Substituting,

$$\frac{1}{f(\lambda)} = k_1\,(n_1(\lambda) - 1) + k_2\,(n_2(\lambda) - 1) - \frac{d}{\frac{1}{k_1\,(n_1(\lambda) - 1)}\;\frac{1}{k_2\,(n_2(\lambda) - 1)}}$$

- For the two focal lengths f(λ_R) and f(λ_B) to be equal, one must place lenses at a distance given by solving for d:

$$d = \frac{1}{k_1 k_2}\,\frac{k_1\,(n_1(\lambda_B) - n_1(\lambda_R)) + k_2\,(n_2(\lambda_B) - n_2(\lambda_R))}{(n_1(\lambda_B) - 1)\,(n_2(\lambda_B) - 1) - (n_1(\lambda_R) - 1)\,(n_2(\lambda_R) - 1)}$$

- If lenses touch, d=0:

$$\frac{k_1}{k_2} = -\frac{n_2(\lambda_B) - n_2(\lambda_R)}{n_1(\lambda_B) - n_1(\lambda_R)}$$

- Now we can have focal length of yellow light ($\lambda_Y \approx \lambda_R + \lambda_B$)/2)

$$\frac{1}{f_1(\lambda_Y)} = k_1\,(n_1(\lambda_Y) - 1)$$

$$\frac{1}{f_2(\lambda_Y)} = k_2\,(n_2(\lambda_Y) - 1)$$

- which is

$$\frac{k_1}{k_2} = \frac{n_2(\lambda_Y) - 1}{n_1(\lambda_Y) - 1}\,\frac{f_2(\lambda_Y)}{f_1(\lambda_Y)}$$

Chromatic aberrations

- So, we have:

$$\frac{f_2(\lambda_Y)}{f_1(\lambda_Y)} = \frac{(n_2(\lambda_B) - n_2(\lambda_R)) / (n_2(\lambda_Y) - 1)}{(n_1(\lambda_B) - n_1(\lambda_R)) / (n_1(\lambda_Y) - 1)} = \frac{w_2}{w_1}$$

  where w_1,w_2 are the dispersive powers associated with the refraction indexes n_1,n_2.
- Take the standardized Fraunhofer spectral lines F,D,C and the wavelengths

$$\lambda_F = 486.1 \text{ nm}$$
$$\lambda_D = 589.2 \text{ nm}$$
$$\lambda_C = 656.3 \text{ nm}$$

- We can now define dispersive power of an optical material w:

$$V = \frac{n_D - 1}{n_F - n_C}$$

  where V=1/w and is called *Abbe number*, or *dispersive index*.
- Here, n_D=n(lD), .
- For lenses, is desirable to have materials with low dispersion, or high Abbe numbers.

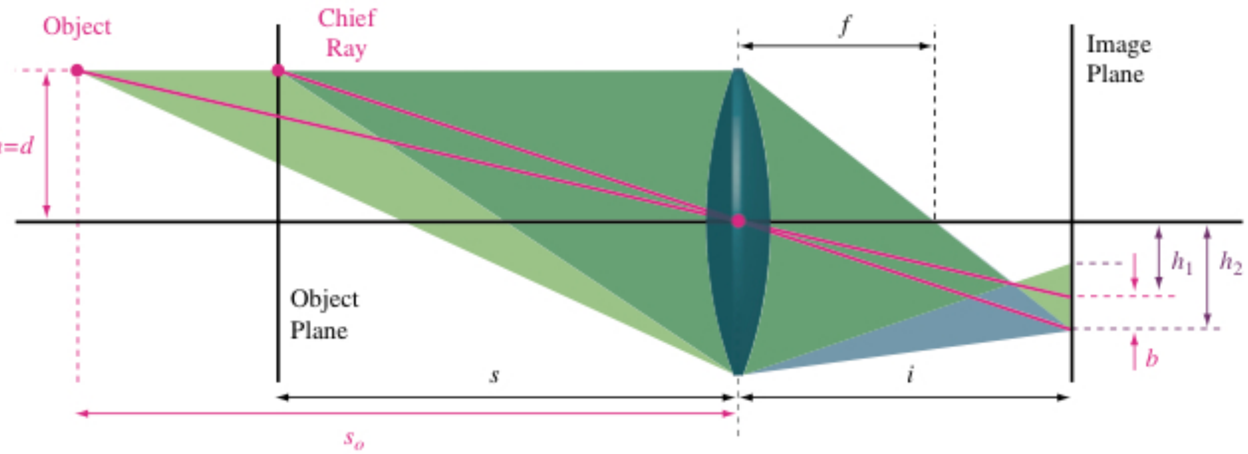
Blur circle

- An ideal optical system would image a point source onto a single point on the image plane.
- Due to aberrations: blurred shape on the image plane
  - Can be approximated as a circle: its radius can be approximated as follows:
  - Place object at same height h of lens aperture height
  - Radius b of blur circle is:

$$b = h_2 - h_1 = di\left(\frac{1}{s} - \frac{1}{s_o}\right) = di\frac{s_o - s}{s\,s_o}.$$

  - From Gaussian lens formula

$$\frac{1}{f} = \frac{1}{s} + \frac{1}{i}$$



- Rewriting: $i = \frac{s f}{s - f}$
- Thus:

$$b = \frac{s f d}{s - f}\frac{s_o - s}{s\,s_o} = \frac{p f d}{s - f}$$

where $p(s_0\text{-}s)/ss_0$ can be seen as percentage focus error.

Depth of field

- Points on image plane have maximum sharpness.
- Objects are not all on image plane: some before, some after.
- There is a region in which points are focused reasonably sharp: *depth of field*.
- Sharpness depends on:
  - Size of image plane
  - Sensor resolution
  - Image reproduction size
  - Angular resolution of human visual system

Depth of field

- If we can make sure that a circle of radius b leads to an image that in the reproduction appears as a single point, then, assuming
  - Camera focused at distance s_0
  - Blur circle smaller aperture ($b \ll d$)
- Then distance of lens to nearest point s_{near} of acceptable focus is

$$s_{\text{near}} = \frac{s_o f}{f + \frac{b}{d}\,(s_0 - f)}$$

and for farthest point it is

$$s_{\text{far}} = \frac{s_o f}{f - \frac{b}{d}\,(s_o - f)}$$

- The global depth of field is then

  s_{far}-s_{near}

- Far plane becomes infinite when denominator goes to 0, i.e. when

$$f = \frac{b}{d}\,(s_o - f)$$

solve for s_0: *hypertocal distance*

$$s_o = f\left(\frac{d}{b} + 1\right)$$

which corresponds to the near plane:

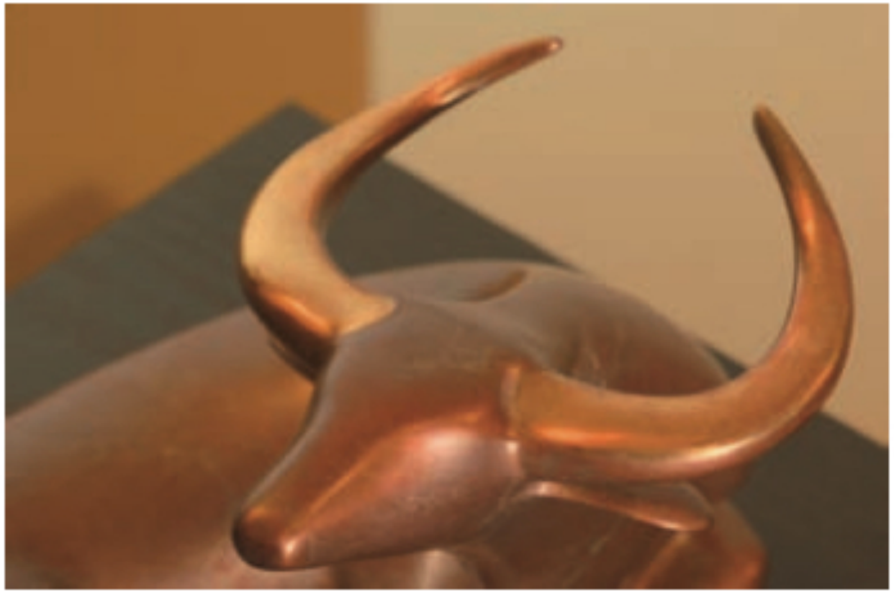
$$s_{\text{near}} = \frac{f}{2}\left(\frac{d}{b} + 1\right)$$

so, if the camera fucused on the hyperfocal plane, all objects between the near plane and infinity will be in focus

- Notice that the aperture *d* affects the depth of field!

Depth of field

f/3.2



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