Computer Animation
3-Interpolation
SS 16

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Parametric curves

- Curves and surfaces can have explicit, implicit, and parametric representations.
  - Explicit equations are of the form $y=f(x)$
  - Implicit equations of the form $f(x,y)=0$
  - Parametric equations are of the form

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

- Parametric representations are the most common in computer graphics and animation.
- They are independent from the axes

Parametric curves

- Parametrization is not unique: take a look at the straight line:
  $L(P_0,P_1) = P_0 + u(P_1-P_0)=$
  $(1-u)P_0+uP_1,$
  $u\in[0,1]$
  $L(P_0,P_1) = v(P_1-P_0)/2 +$
  $(P_1+P_0)/2,\ v \in [-1,1]$
- They represent the same line

- Parameterizations can be changed to lie between desired bounds.
  To reparameterize from $u\in[a,b]$ to $w\in[0,1]$,
  we can use
  $w=(u-a)/(b-a)$, which gives $u = w(b-a) + a$.
- Thus, we have:

  $P(u),\ u\in[a,b] =$
  $P(w(b-a)+a),\ w\in[0,1]$

Linear interpolation

- Consider the straight line passing through P_0 and P_1:
  $P(u)=(1-u)P_0+uP_1$
- Since (1-u) and u are functions of u, one can rewrite the eq. above as
  $P(u)=F_0(u)P_0+F_1(u)P_1$
- Note that $F_0(u)+F_1(u)=1$
- $F_0(u)$ and $F_1(u)$ are called *blending functions*.

- Alternatively, one can rewrite the function as
  $P(u)=(P_1- P_0)u+P_0$
  $P(u)=\underline{a}_1u+\underline{a}_0$
- This called the algebraic form of the equation

Linear interpolation

- One can also rewrite theese equations in matrix notation:

$$P(u) = \begin{bmatrix} F_0(u) \\ F_1(u) \end{bmatrix} \begin{bmatrix} P_0 & P_1 \end{bmatrix} = FB^T$$

$$P(u) = \begin{bmatrix} u & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$$

$$P(u) = \begin{bmatrix} u & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \end{bmatrix} = U^T MB = FB = U^T A$$

- Note that the last one of these equations decomposes the equation in the product of variables (U), coefficients (M) and geometric information (B)

Arc length

- Note that there is not necessarily a linear relation between the parameter u and the arc length described by the curve
- For example, also the equation

  $$P(u)=P_0+((1-u)u+u)(P_1-P_0)$$

  represents the same straight line, but the relationship between u and the arc length is non linear.
- This means that there is not necessarily an obvious relationship between changes in parameter and distance travelled and changes in the parameter

Derivatives of a curve

- Any parametric curve of polynomial order can be expressed in the form
  $P(u)=U^TMB$
- Since only the matrix U contains the variable, then it is easy to compute the derivative of a parametric curve
- For a curve of third degree we have

$P(u)=U^TMB=$
$[u^3\ u^2\ u\ 1]\ MB$

$P'(u)=U'^TMB=$
$[3u^2\ 2u\ 1\ 0]\ MB$

$P''(u)=U''^TMB=$
$[6u\ 2\ 0\ 0]\ MB$

Hermite interpolation

- Hermite interpolation generates a cubic polynomial between two points.
- Here, to specify completely the curve the user needs to provide two points P_0 and P_1 and the tangent to the curve in these two points P'_0 P'_1
- Remember, we write in the form $P(u)=U^TMB$

- For Hermite interpolation we have

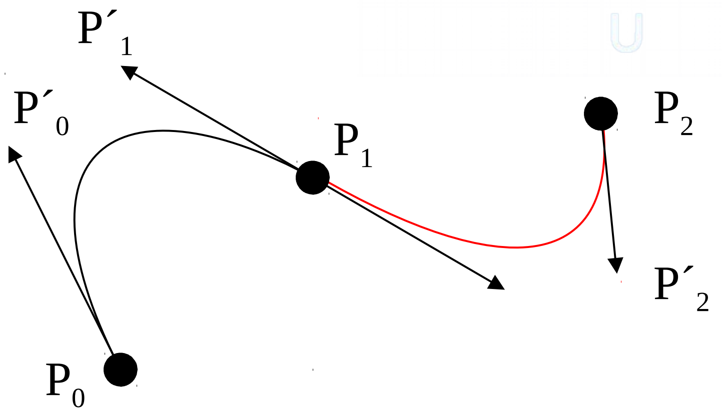
$$U^T \lfloor u^3 \quad u^2 \quad u \quad 1 \rfloor$$

$$M = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} P_0 \\ P_1 \\ P'_0 \\ P'_1 \end{bmatrix}$$

Hermite interpolation

- Suppose that an interpolation curve is wanted passing through n points $P_0, P_1, \ldots, P_n$.
- The interpolation curve through them can be defined as a piecewise defined curve



- In fact, if one ensures that the resulting curve is not only continuous at the joints, but also that
  - Its tangent (=velocity)
  - Its second order derivative (= acceleration)

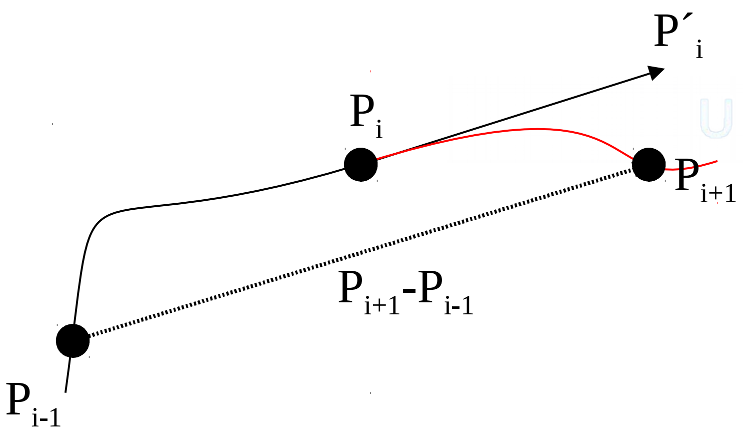
  are continuous, then the curve can be used also in animations.

Continuity: parametric and geometric

- For a piecewise defined curve, there are two main ways of defining the continuity at the borders of the single intervals of definition
  - 1st order parametric continuity (C^1): the end tangent vector at the two ends must be exactly the same
  - 1st order geometric continuity (G^1): the direction of the tangent must be the same, but the magnitudes may differ
  - Similar definitions for higher oder continuity (C^2-G^2)
- Parametric continuity is sensitive to the „velocity“ of the parameter on the curve, geometric continuity is not

Catmull-Rom spline

- A Catmull-Rom spline is a special Hermite curve where the tangent of the middle points is computed as one half the vector joining the previous control point to the next one



- $P'_i = 1/2(P_{i+1} - P_{i-1})$
- From this we deduce:

$$U^T = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix}$$

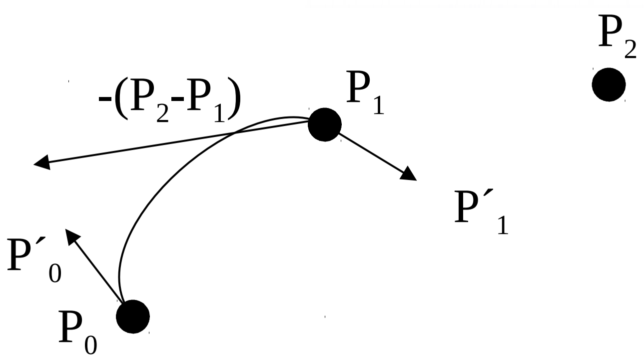
$$M = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 2 & -5 & 4 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{bmatrix}$$

Catmull-Rom spline

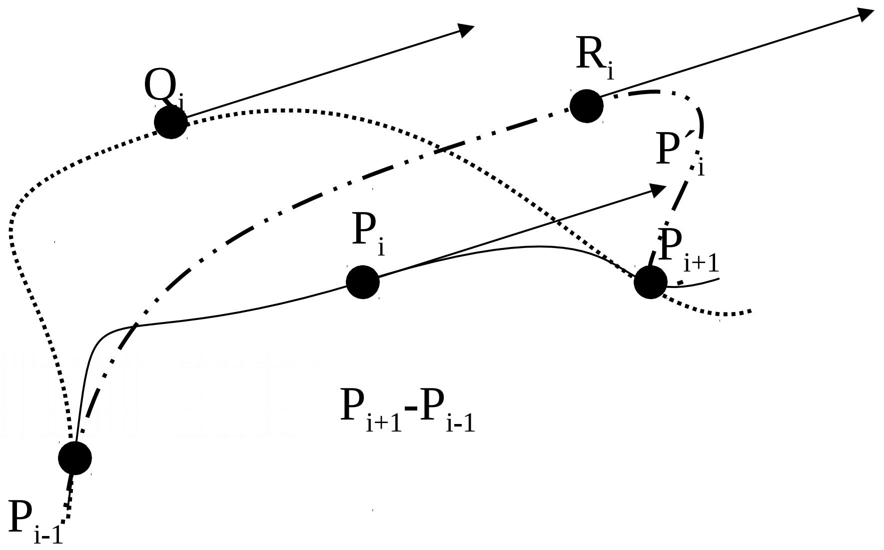
- If one wants to write the complete Catmull-Rom spline, one needs a method to find the tangents at the initial and final points

- One method used involves subtracting P_2 from P_1 and then using the point obtained as the direction of the tangent
- $P'_0 = ½(P_1-(P_2-P_1)-P_0) = ½(2P_1-P_2-P_0)$



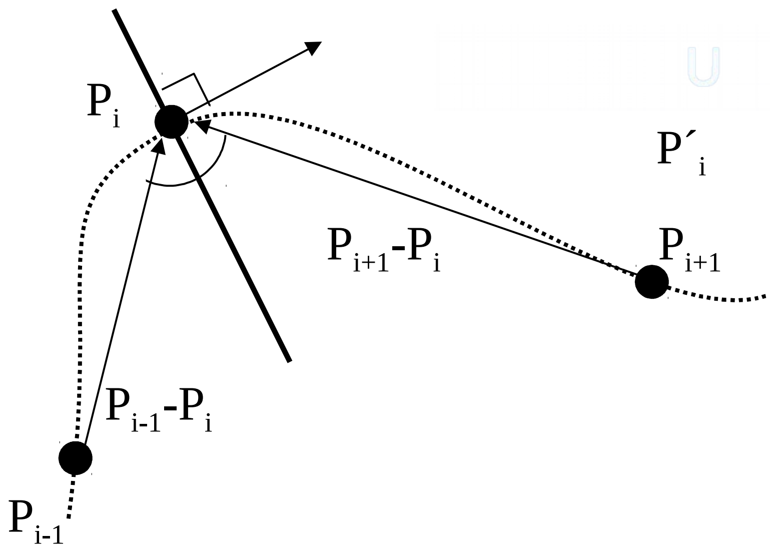
Catmull-Rom spline

- Advantage of Catmull-Rom splines: fast and simple computations
- Disadvantage: tangent vector is not this flexible: for example, all curves below have same tangent in P_i



Catmull-Rom spline

- A simple alternative is to compute the tangent at the point as the ⊥ of the bisector of the angle formed by $P_{i-1}-P_i$ and $P_{i+1}-P_i$



- Another modification is to not impose same tangent length at the points, but different lenghts on the two sides of the joint.
- The tangent vectors can be scaled for example by the ratio of the distance between current point and former point and the distance between former and next point.
- This obtains more „adaptable“ tangents, but trades also off C^1 continuity

Four point form

- Suppose you have 4 points $P_0P_1P_2P_3$ and to want a cubic segment fitting through them.
- Une can set up a linear system of equations through the points and solve

$$P(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \\ m_{30} & m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} u_0^3 & u_0^2 & u_0 & 1 \\ u_1^3 & u_1^2 & u_1 & 1 \\ u_2^3 & u_2^2 & u_2 & 1 \\ u_3^3 & u_3^2 & u_3 & 1 \end{bmatrix} \begin{bmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \\ m_{30} & m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

Four point form

- In the case that you want the parameter values at the points to be (0,1/3,2/3,1), the matrix is

$$M = \frac{1}{2}\begin{bmatrix} -9 & 27 & -27 & 9 \\ 18 & -45 & 36 & -9 \\ -11 & 18 & -9 & 2 \\ 2 & 0 & 0 & 0 \end{bmatrix}$$

With this form it is difficult to join segments with C^1 continuity

Blended parabolas

- One other method is through blending two overlapping parabolas
- The blending is done by taking the first 3 points to define a parabola, then the 2nd, 3rd and 4th point to define a second parabola, and then linearly interpolate the parabolas
- This is the resulting matrix for equally spaced points in parametric space

$$M = \frac{1}{2}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 2 & -5 & 4 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

Bezier curves

- Another way of defining a curve is to define it through two endpoints, which are interpolated, and two interior points, which control the shape.
- Bezier curves use the two additional control points to define the tangent
- $P'(0)=3(P_1-P_0)$

  $P'(1)=3(P_3-P_2)$

- The corresponding matrix will be

$$M = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

which corresponds to the basic functions UM

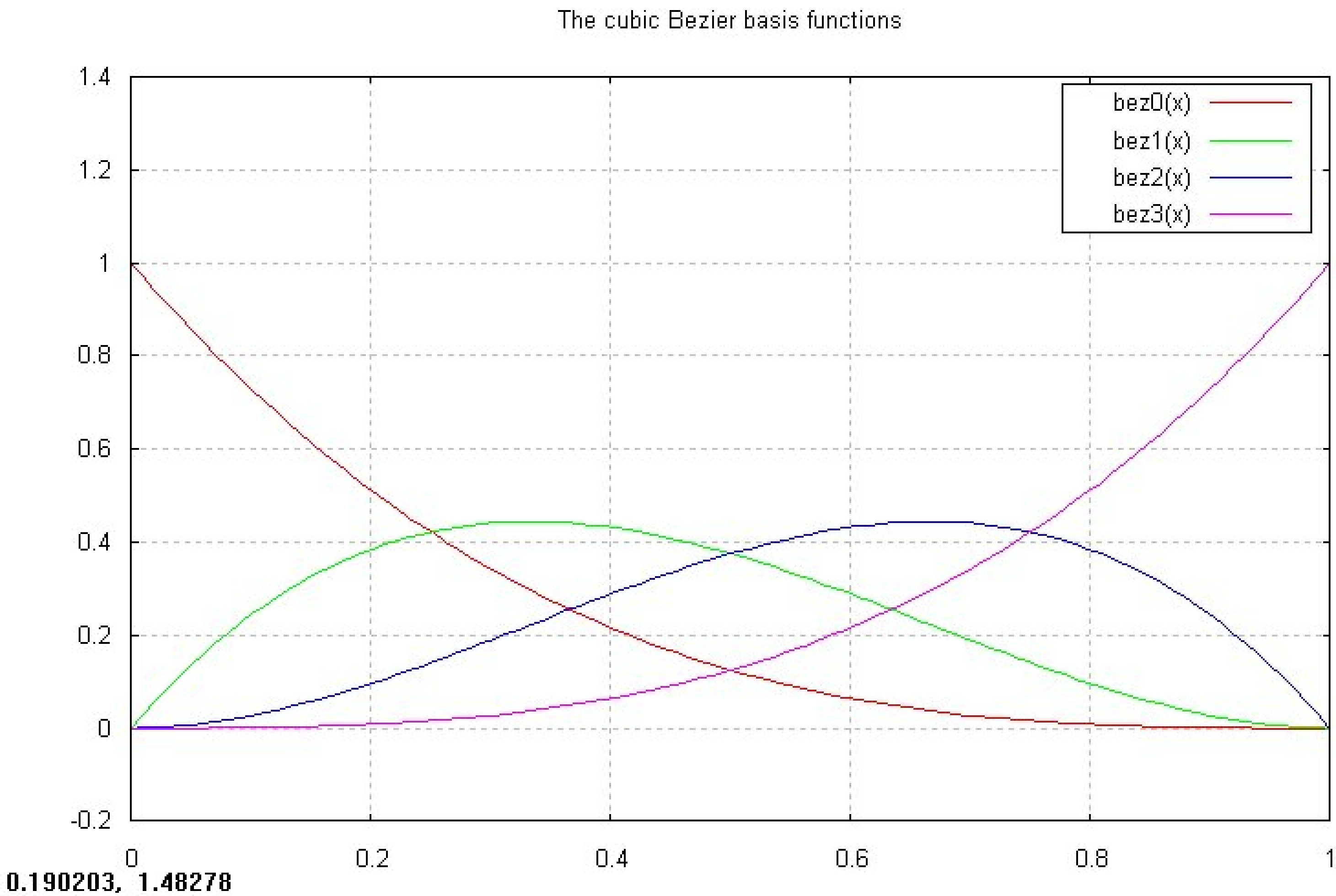
$B_0(t)=(1-t)^3$

$B_1(t)=3t(1-t)^2$

$B_2(t)=3t^2(1-t)$

$B_3(t)=t^3$

Bezier curves



Bezier curves

- In fact, Bezier curves can be of any order. The basis functions are

$$B_{in}(t)=t^i(1-t)^{n-i}n!/i!/(n-i)!$$

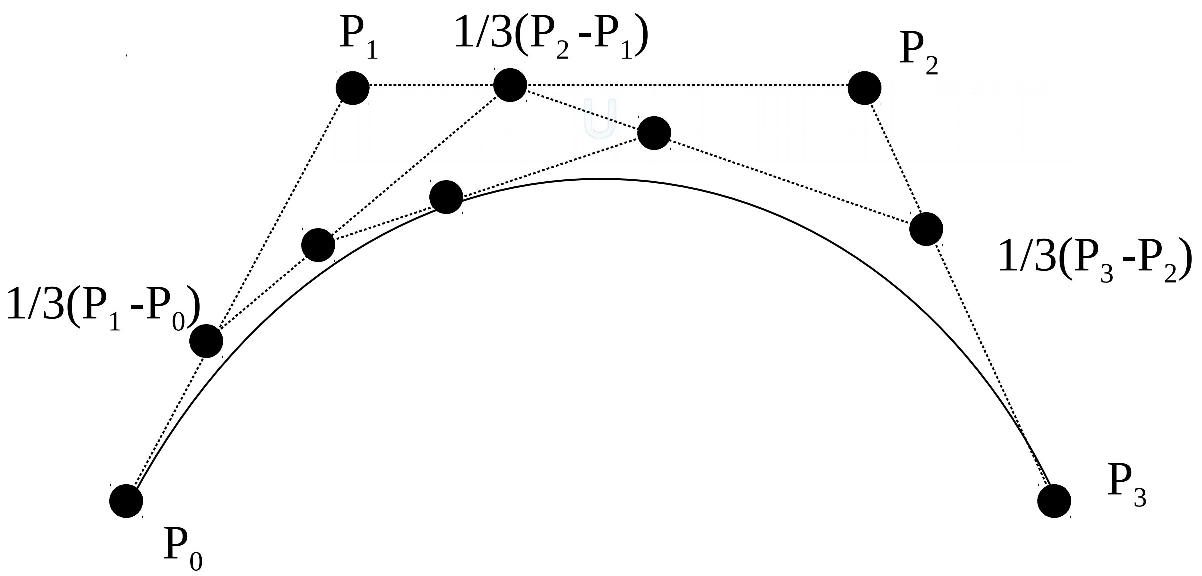
Where n is the degree and $i=0,\ldots,n$.

- And the Bezier curve passing through the points $P_0,P_1,\ldots,P_n$ is

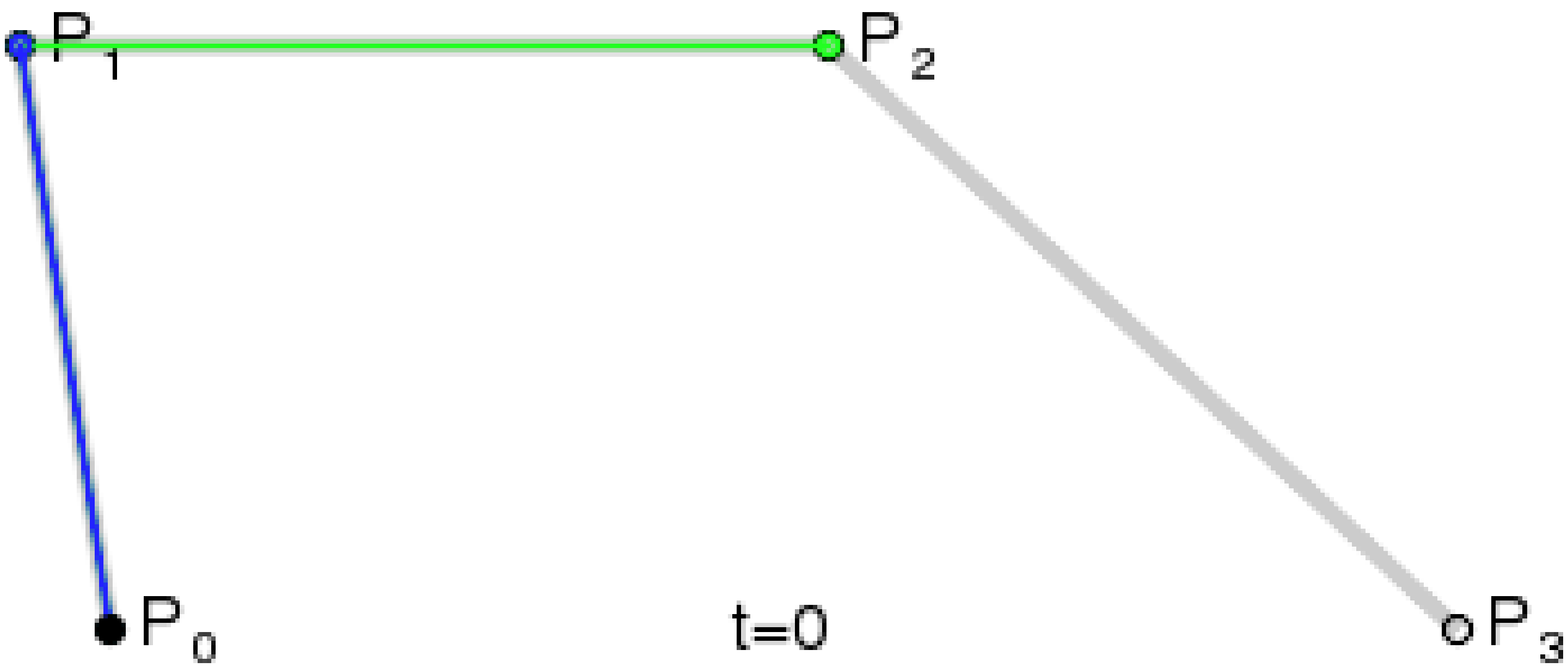
$$Q(T)=\Sigma_{i=0,\ldots,n}B_{in}(t)P_i$$

Bezier curves: De Casteljeau construction

- De Casteljeau came up with a geometric method for constructing a Bezier Curve
- The figure illustrates the construction of a point at t=1/3 for a curve of 3rd degree



Bezier splines



Or check https://www.jasondavies.com/animated-bezier/

Uniform B-splines

- Uniform B-splines are most flexible type of curves, and also more difficult to understand
- They detach the order of the resulting polynomial from the number of control points. Suppose we have a number N of control points.
- Bezier curves are a special case of B-splines
- One starts by defining a uniform knot vector [0,1,2,...,N+k-1], where k is the degree of the B-spline curve and n the number of control points.
- Knots are uniformly spaced.
- If k is the degree of the B-spline, then each single component of the B-spline will be defined between the consecutive control points $P_i, P_{i+1}, .., P_{i+k}$.
- The next bit will be defined between $P_{i+1}, P_{i+2}, .., P_{i+k+1}$

Uniform B-splines

- The equation for *k*-order B-spline with $N+1$ control points $(\boldsymbol{P}_0 , \boldsymbol{P}_1 , \dots , \boldsymbol{P}_N)$ is

  $$\boldsymbol{P}(t) = \Sigma_{i=0,\dots,N} N_{i,k}(t) \boldsymbol{P}_i ,$$
  $$t_{k-1} \leq t \leq t_{N+1}$$

- In a B-spline each control point is associated with a basis function $N_{i,k}$ which is given by the recurrence relations

  $$N_{i,k}(t) =$$
  $$N_{i,k-1}(t) (t - t_i)/(t_{i+k-1} - t_i) +$$
  $$N_{i+1,k-1}(t) (t_{i+k} - t)/(t_{i+k} - t_{i+1}),$$
  $$N_{i,1} = \{1 \text{ if } t_i \leq t \leq t_{i+1},$$
  $$0 \quad \text{otherwise} \}$$

- $N_{i,k}$ is a polynomial of order k (degree $k-1$) on each interval $t_i < t < t_{i+1}$.
- k must be at least 2 (linear) and can be not more, than $n+1$ (the number of control points).
- A knot vector $(t_0,t_1,\dots, t_{N+k})$ must be specified. Across the knots basis functions are C^{k-2} continuous.

Uniform B-splines

- B-spline basis functions like Bezier ones are nonnegative $N_{i,k} \geq 0$ and have "partition of unity" property

  $$\Sigma_{i=0,N} N_{i,k}(t) = 1, \quad t_{k-1} < t < t_{n+1}$$

  therefore

  $$0 \leq N_{i,k} \leq 1.$$

- Since $N_{i,k} = 0$ for

  $t \leq t_i$ or $t \geq t_{i+k}$,

  a control point $\boldsymbol{P}_i$ influences the curve only for $t_i < t < t_{i+k}$.

B-splines

- Depending on the relative spaces between knots in parameter spaces, we can have uniform or non-uniform B-splines
- The shapes of the $N_{i,k}$ basis functions are determined entirely by the *relative* spacing between the knots $(t_0 , t_1 , ... , t_{N+k})$.
- Scaling or translating the knot vector has no effect on shapes of basis functions and B-spline.

- Knot vectors are generally of 3 types:
  - *Uniform knot vectors* are the vectors for which

    $$t_{i+1} - t_i = const,$$

    e.g. *[0,1,2,3,4,5]*.
  - *Open Uniform knot vectors* are uniform knot vectors which have k-equal knot values at each end:

    $$t_i = t_0 , \quad i < k$$
    $$t_{i+1}-t_i = const, \ k-1 \leq i < n+1$$
    $$t_i = t_{k+n} , \quad l \geq n+1$$

    eg*[0,0,0,1,2,3,4,4,4](k=3,N=5)*
  - *Non-uniform knot vectors*. This is the general case, the only constraint is the standard $t_i \leq t_{i+1}$.

B-splines

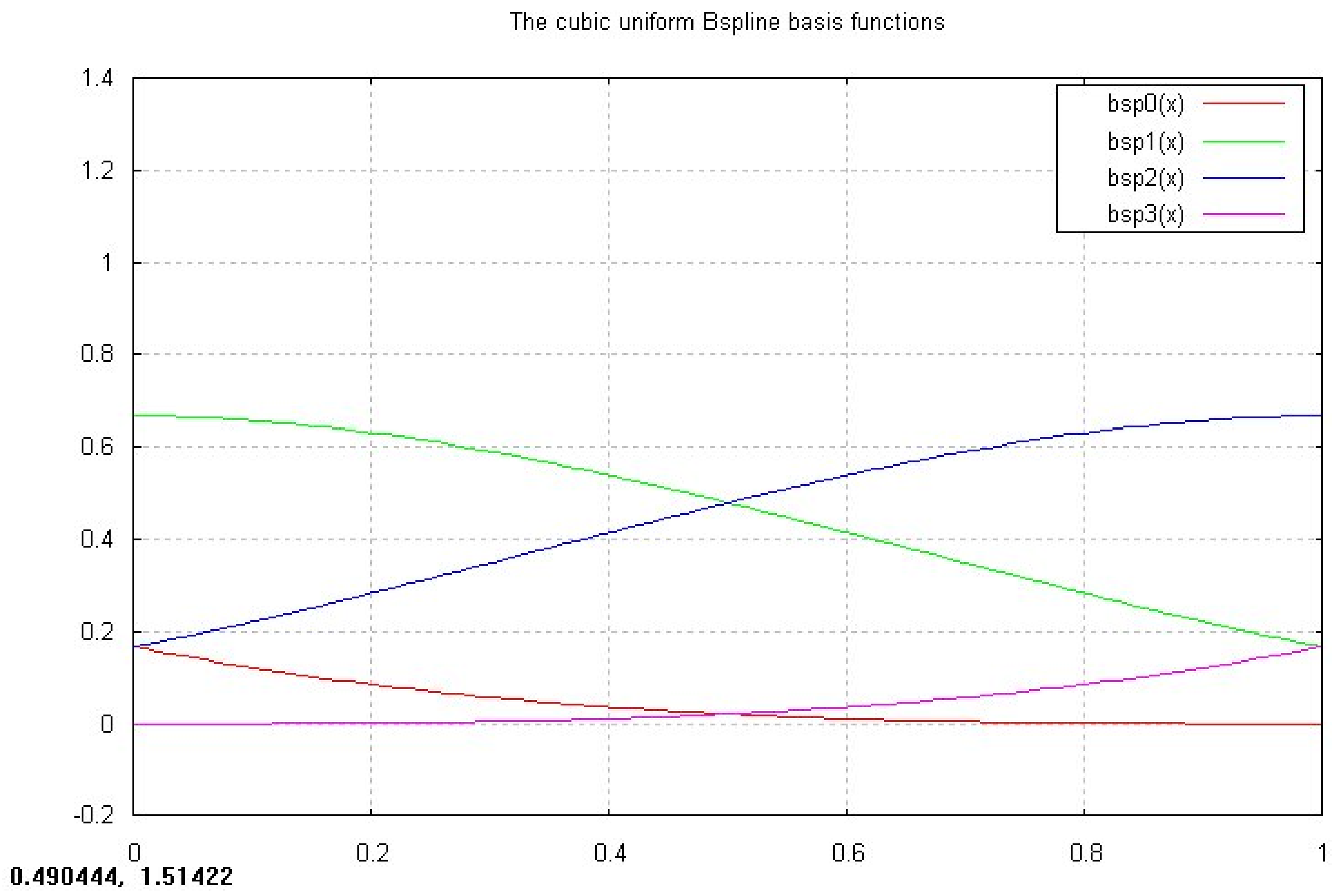
- The main properties of B-splines
  - composed of *(n-k+2)* Bezier curves of *k*-order joined C^{k-2} continuously at knot values $(t_0 , t_1 , ... , t_{n+k})$
  - each point affected by *k* control points
  - each control point affected *k* segments
  - inside convex hull
  - affine invariance
  - uniform B-splines don't interpolate deBoor control points $(\boldsymbol{P}_0 , \boldsymbol{P}_1 , ... , \boldsymbol{P}_N)$

Uniform 3rd order B-splines

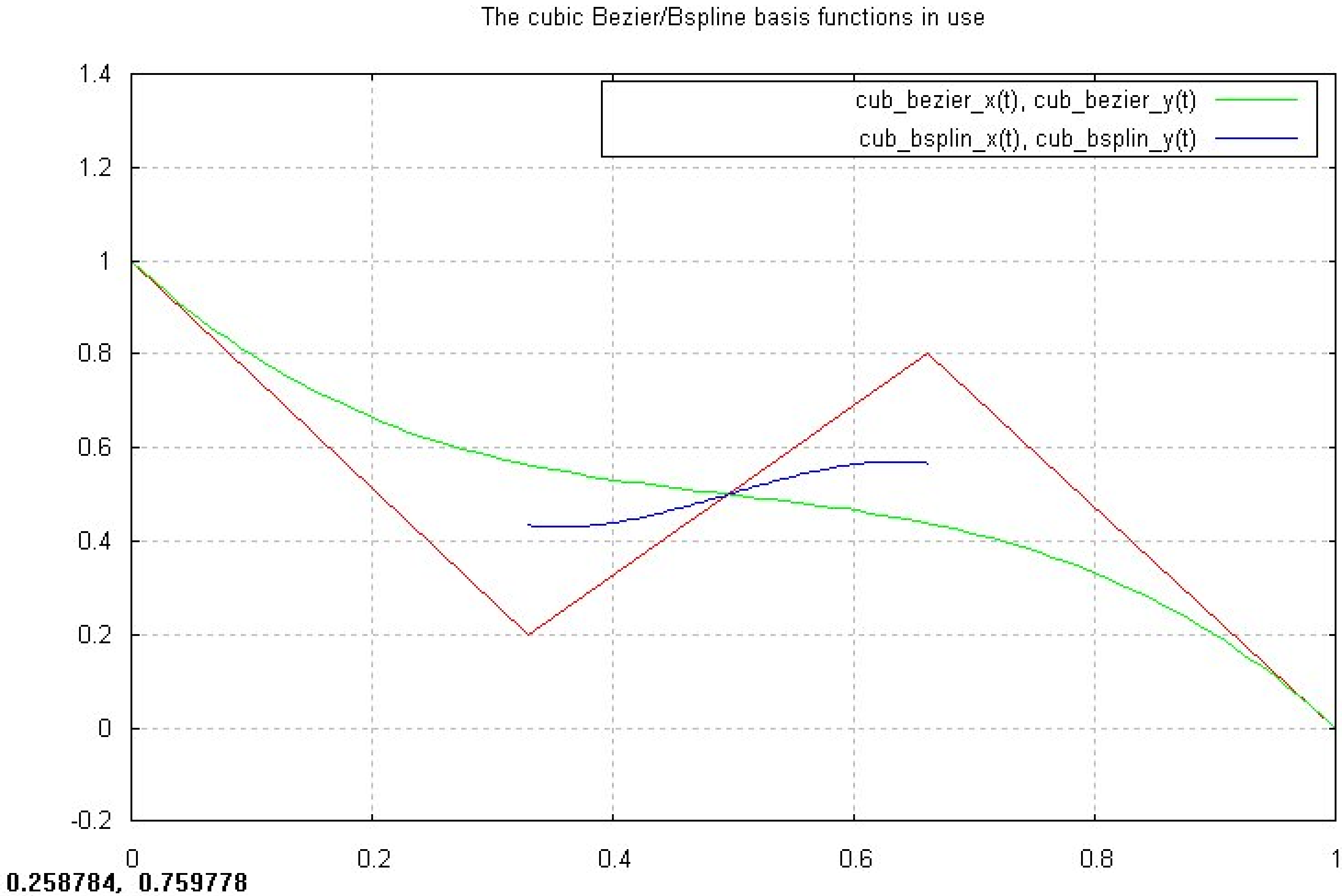
- For a B-spline of order 3, and the four control points P_i, P_{i+1}, P_{i+2}, P_{i+3} we have that the B-spline can be written as
- The curves defined by increasing i=0,...,N-3 will define a C^2-continuous curve

$$P(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \frac{1}{6} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} P_i \\ P_{i+1} \\ P_{i+2} \\ P_{i+3} \end{bmatrix}$$

Uniform B-splines

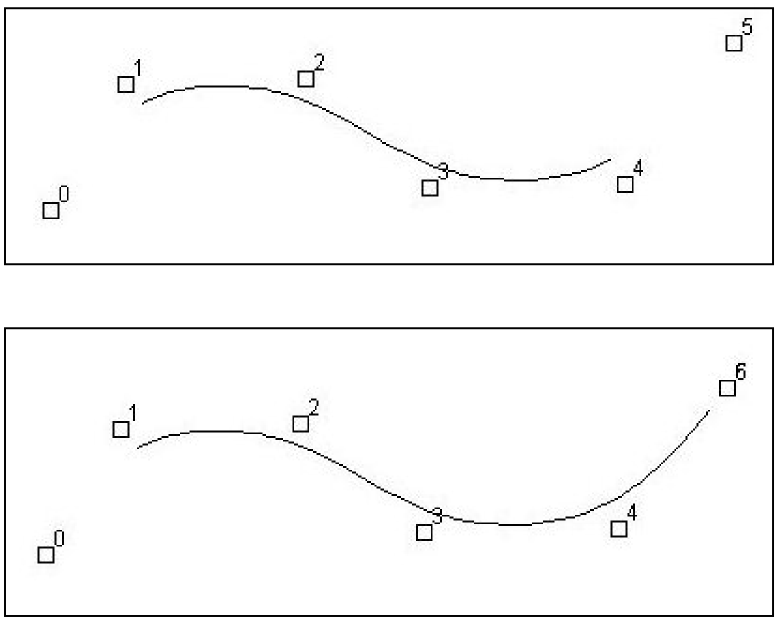


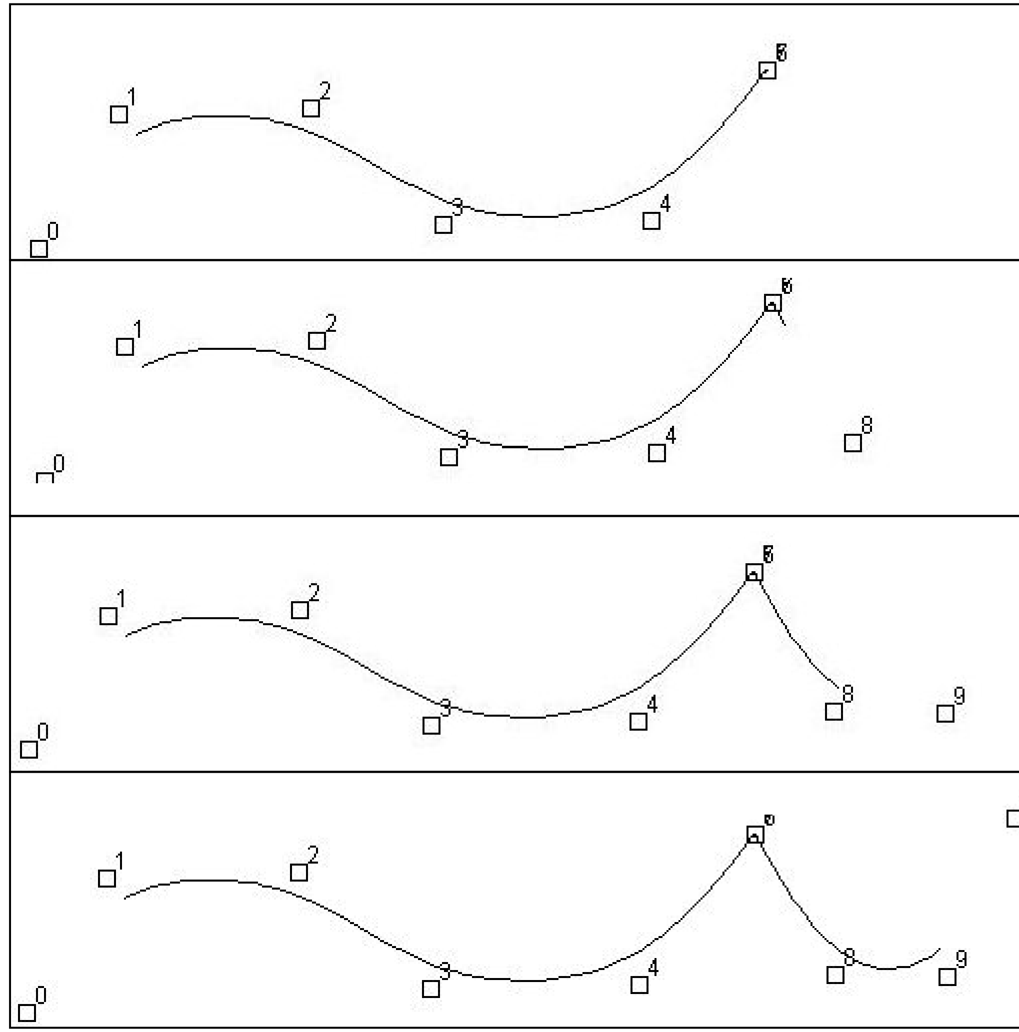
Bezier and B-spline curves



B-splines: multiple knots

- Knots can be made to coincide to obtain cusps and let the curve pass through a desired point

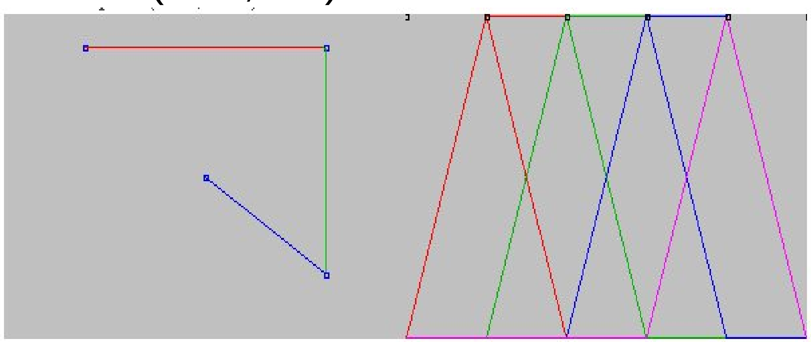




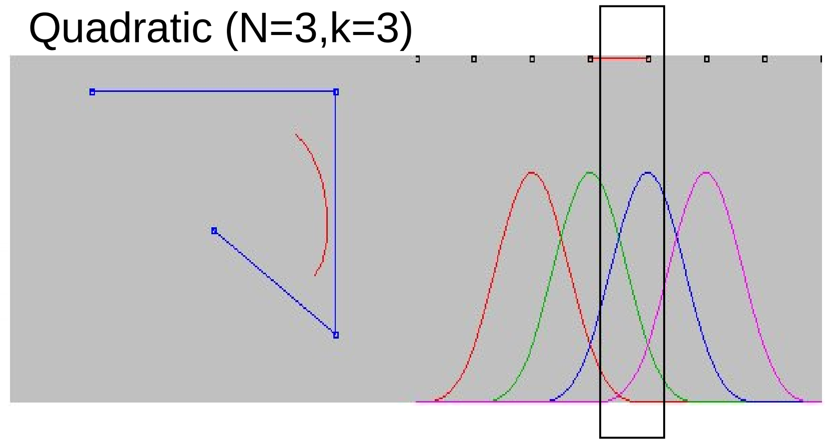
Uniform B-splines: examples

- For a given order *k*, uniform B-splines are shifted copies of one another since all the knots are equispaced

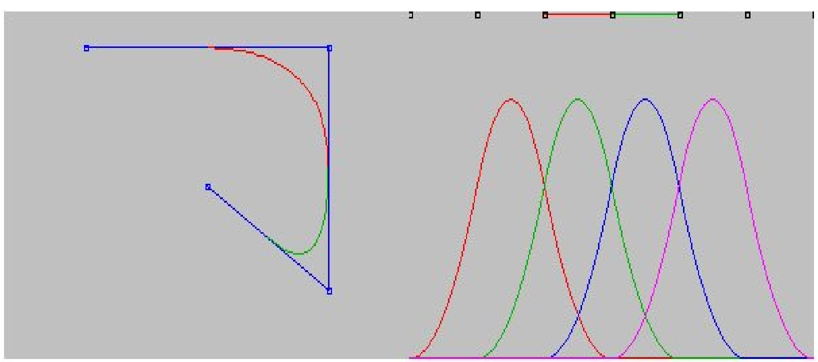
Linear (N=3,k=2)



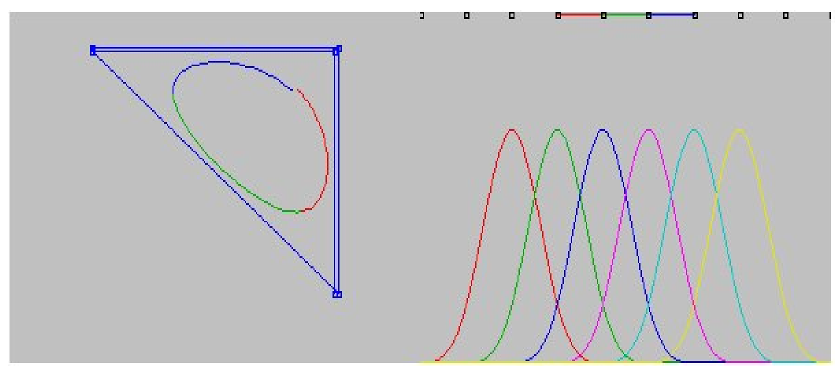
Quadratic (N=3,k=3)



Cubic (N=3,k=4)



Closed (N=5,k=4)



NURBS

- Stands for non-uniform rational B-splines
  - Non-uniform: knots are not at same distance
  - Rational: it‘s a fraction, with B-splines at the numerator and denominator
- Advantages: one can express circular arcs with NURBS
- Disadvantages: lots of computational effort

NURBS

- Recall that the B-spline is weighted sum of its control points

  $$\boldsymbol{P}(t) = \Sigma_{i=0,..,N} N_{i,k}(t)\, \boldsymbol{P}_i\,,$$
  $$t_{k-1} \le t \le t_{N+1}$$

  and the weights $N_{i,k}$ have the "partition of unity" property

  $$\Sigma_{i=0,..,N} N_{i,k}(t) = 1\,.$$

- As weights $N_{i,k}$ depend on the knot vector only, it is useful to add to every control point one more weight w_i which can be set independently

  $$\boldsymbol{P}(t) = \Sigma_{i=0,..,N} w_i N_{i,k}(t) \boldsymbol{P}_i / \Sigma_{i=0,..,N} w_i N_{i,k}(t)\,.$$

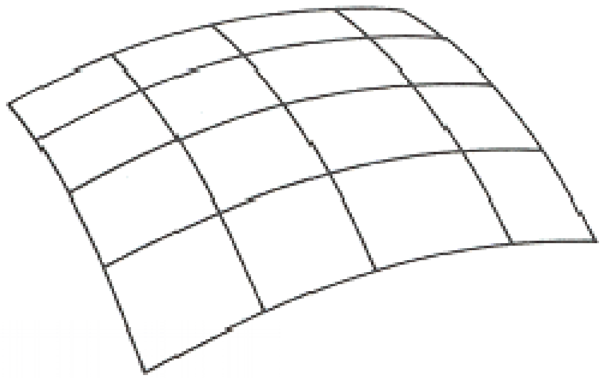
- Increasing a weight w_i makes the point more influence and attracts the curve to it.
- The denominator in the 2nd equation normalizes weights, so we will get the 1st equation if we set $w_i = const$ for all i.
- Full weights $w_i N_{i,k}$ satisfy the "partition of unity" condition again.

Global vs local control

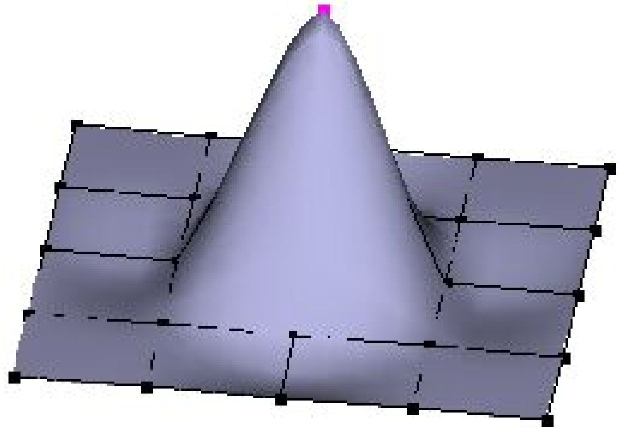
- Depending on the curve formulation, moving a control point can have different effects
  - Local control: in this case the effect of the movement is limited in its influence along the curve
  - Global control: moving a point redefines the whole curve
- Local control is the most desirable for manipulating a curve
- Almost all of the piecewise defined curves have local control
- Only exception: Hermite curves enforcing C^2 continuity

Modeling with splines

- 3D Splines can be used to represent object boundaries by piecewise defined „patches“ joined at their definition edges so that they are continuous at the joins, like a „patchwork“
- Splines are very flexible in shape modeling
- But what is behind spline patches?



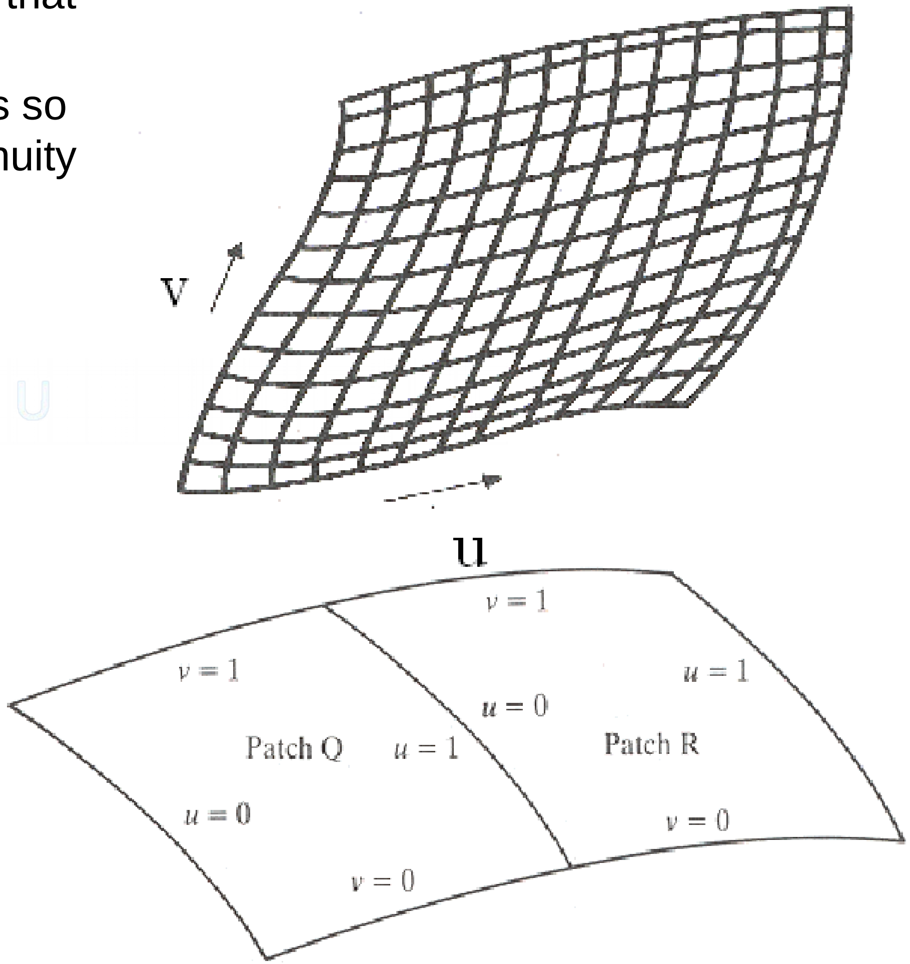
Courtesy T. Funkhouser, Princeton University



Courtesy Russian Academy of Sciences

Spline patches

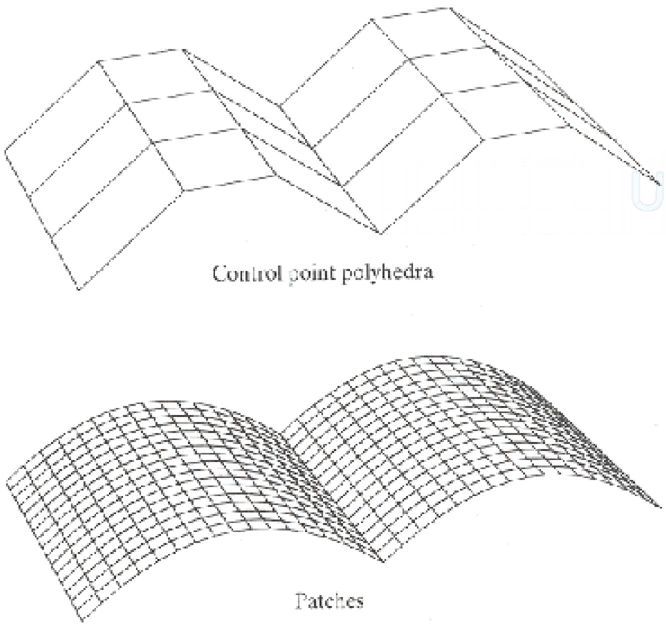
- Here the idea is to find families of piecewise parametric functions that allow a good control on shape
- Patches are joined at the edges so as to achieve the desired continuity
- Each patch is represented in parametric space



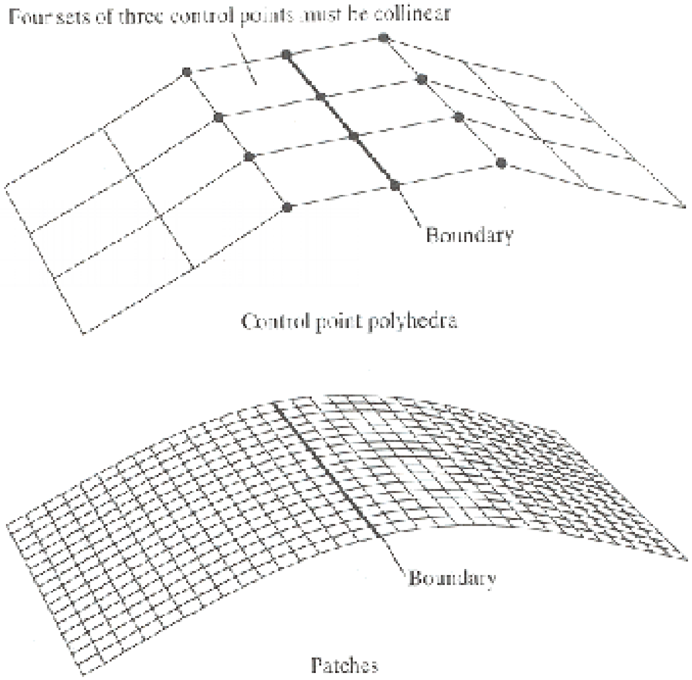
Courtesy T. Funkhouser, Princeton University

Spline patches

- C^0 continuity



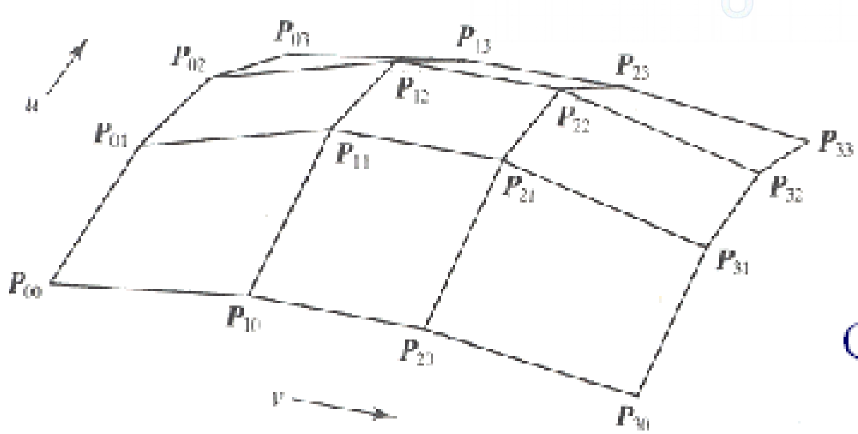
- C^1 continuity

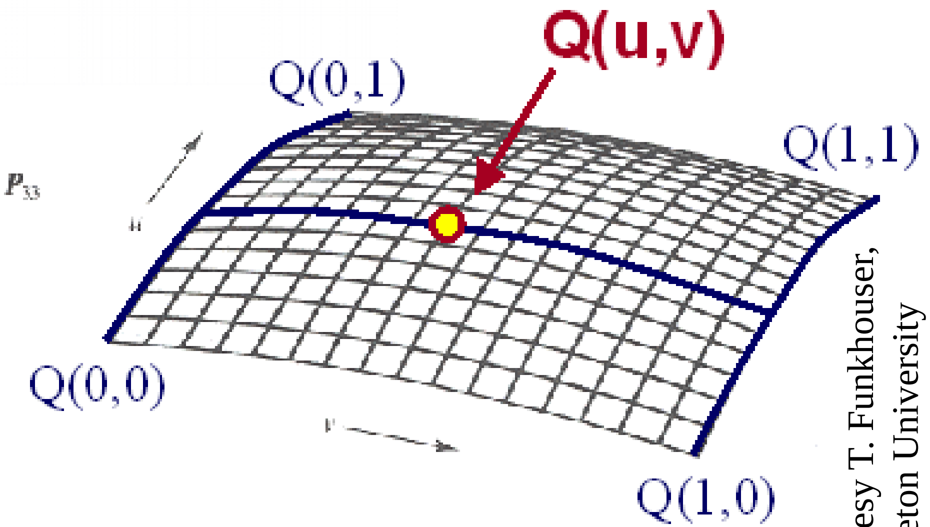


Courtesy T. Funkhouser,
Princeton University

Spline patches

- A point Q on a patch is the tensor product of parametric functions defined by control points





Courtesy T. Funkhouser, Princeton University

Spline patches

- A point Q on any patch is defined by multiplying control points by polynomial blending functions

$$Q(u,v) = UM \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \end{bmatrix} M^T V^T$$

$$U = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix}$$

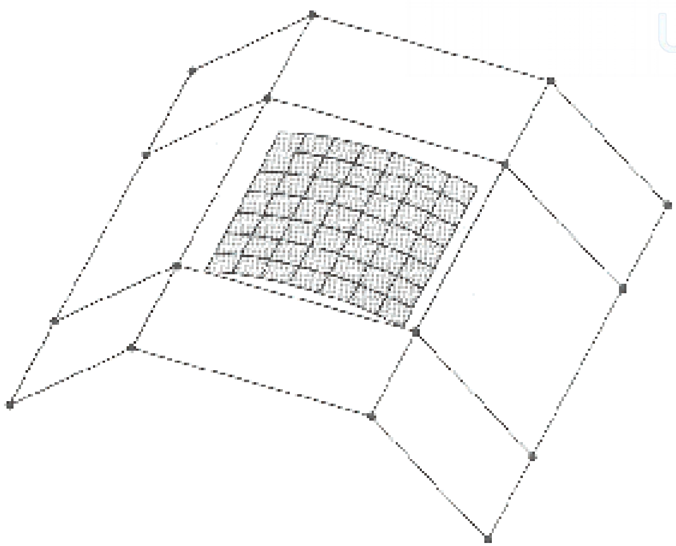
$$V = \begin{bmatrix} v^3 & v^2 & v & 1 \end{bmatrix}$$

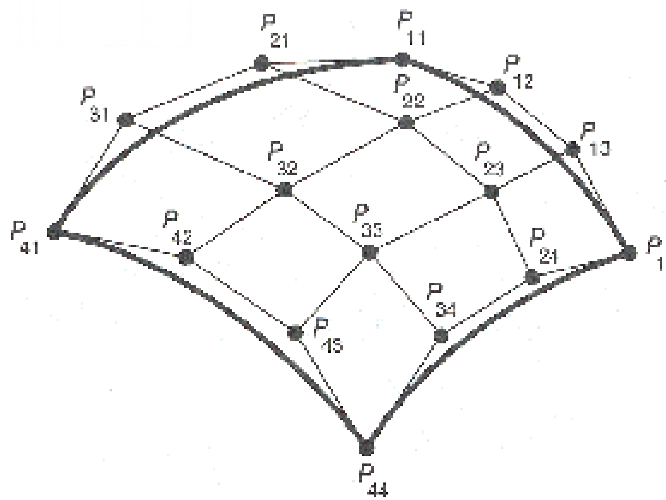
- What about M then? M describes the blending functions for a parametric curve of third degree

Spline patches

$$M_{B\text{-}spline} = \begin{bmatrix} -1/6 & 1/2 & -1/2 & 1/6 \\ 1/2 & -1 & 1/2 & 0 \\ -1/2 & 0 & 1/2 & 0 \\ 1/6 & 2/3 & 1/6 & 0 \end{bmatrix}$$

$$M_{Bezier} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

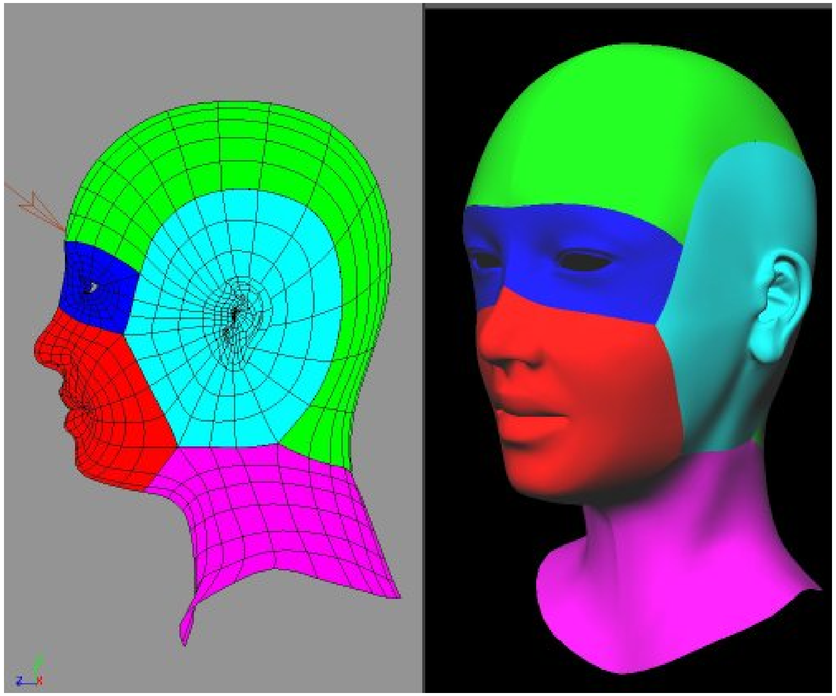




Courtesy T. Funkhouser,
Princeton University

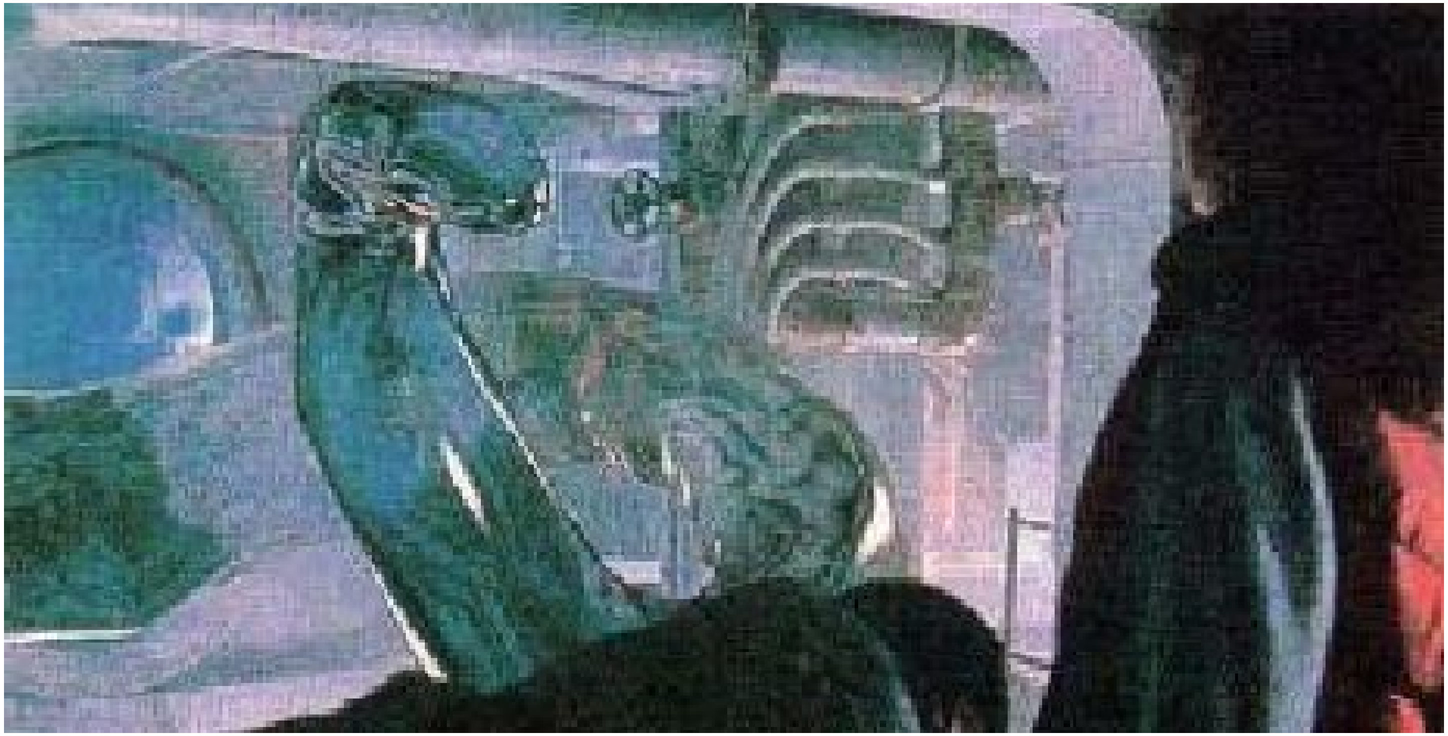
Spline patches

- Third order patches allow the generation of free form surfaces, and easy controllability of the shape
- Why third order functions?
  - Because they are the minimal order curves allowing inflection points
  - Because they are the minimal order curves allowing to control the curvature (= second order derivative)



Courtesy Softimage Co.

End



+++ Ende - The end - Finis - Fin - Fine +++ Ende - The end - Finis - Fin - Fine +++