Fundamentals of Imaging
Geometrical optics

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This slide pack

• In this part, we will introduce geometrical optics:
  – Principles of geometrical optics
  – Fermat’s principle
  – Perspective-projective geometry
  – Optical systems
    • Optical image formation
  – Absolute instruments
  – Imaging geometry
  – Imaging radiometry
  – On-axis and off-axis irradiance
  – Effects: Vignetting, glare
Image capture

- Imaging:
  - mapping of some characteristics of the real world (object space)
  - into another representation of this space (image space)
- In general, a capturing system will be composed of several components
  - Components are optimized to convey light to the sensing device
  - Several variables are available here, and they affect the quality of the system
- Despite knowing that light is generated by quantum mechanics
- In general one would use the geometric (optical) representation of light for this
- Main assumption:
  - Light can be treated as rays, because its wavelength is less than 1 micron
  - Neglectable with respect to distances travelled
  - Characteristics can be studied geometrically
  - Whenever light has to be treated as waves, one has to do it explicitly
The basis of geometrical optics

• An arbitrary complex time function of the electromagnetic field can be decomposed into Fourier components of time harmonics.

• Let us take a general time harmonic field:
  \[ \mathbf{E}(r, t) = \mathbf{E}_0(r) e^{-i\omega t} \]
  \[ \mathbf{H}(r, t) = \mathbf{H}_0(r) e^{-i\omega t} \]
  in regions free of currents and charges, \( \mathbf{E}_0 \) and \( \mathbf{H}_0 \) will satisfy time-free Maxwell equations.

• Define \( k_0 = \frac{2\pi}{\lambda_0} \), where \( \lambda_0 \) is the wavelength in vacuum.

• Away from the source, the fields can be represented as general fields
  \[ \mathbf{E}_0(r) = \mathbf{e}(r) e^{-i k_0 \psi(r)} \]
  \[ \mathbf{H}_0(r) = \mathbf{h}(r) e^{-i k_0 \psi(r)} \]

• Assuming that \( \lambda_0 \rightarrow 0 \), and that terms containing \( 1/k_0 \) can be neglected, from Maxwell’s equation one can derive

\[ \nabla \psi \cdot \nabla \psi = \left( \frac{\partial \psi}{\partial x} \right)^2 + \left( \frac{\partial \psi}{\partial y} \right)^2 + \left( \frac{\partial \psi}{\partial z} \right)^2 = n^2(x, y, z) \]

eikonal equation

\( n \): index of refraction

\( \psi \): eikonal function

nabla operator \( \nabla = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right) \)

• Where \( \psi \) constant phases are constant (geometrical wavefronts)

• Energy of the electromagnetic wave propagates with velocity \( v = c/n \) in the surface normal to the wavefronts

• Thus light rays are orthogonal to the geometrical wavefronts

(1) In this chapter, bold variables will represent vectors
The basis of geometrical optics

• Let
  – \( r(s) \) position vector of a point on a light ray,
  – \( s \) arc length of ray,
  – Then \( \frac{dr}{ds} \) is a unit vector pointing to the direction of the light ray

• One can then rewrite the eikonal equation as
  \[
  n \frac{dr}{ds} = \nabla \psi
  \]

• Because the distance between two neighbouring wavefronts \( d\psi \) can be expressed as
  \[
  d\psi = dr \cdot \nabla \psi = nds
  \]

  the integral \( \int_{P_1}^{P_2} n \, ds \)

  taken on a curve along the path from \( P_1 \) to \( P_2 \) is called the optical path length between the points

• In most cases, the light ray travels along the path of shortest optical length

• However, this is not always true:
  – Light rays travel along the path that have zero derivative with respect to time or with respect to the optical path length (Fermat’s principle)

• Because the light ray is gradient of a scalar field, then if the ray vector is operated by a curl operator, the result is zero

• This proves Snell’s law: incident ray, refracted ray and surface normal are all in the same plane
Fermat’s principle

- Eikonal equation describes geometrical optics
- Alternatively, one can use Fermat’s principle: light follows a ray such that optical path length is an extremum
- Optical path length: \( \int_{a}^{b} n \, ds \)
  - ds: arc length
  - n: refraction index
  - a,b: start and end of path
- Minimizing this integral through variation calculus results in the ray equation
  \[
  \frac{d}{dl} \left( n(r) \frac{dr}{dl} \right) = \nabla n(r)
  \]
- Meaning:
  - at every point of the medium, tangent and normal of a ray form a plane, called osculating plane
  - The gradient of the refracting index must lie in this plane
- Valid for inhomogeneous isotropic media which are stationary over time
- A consequence of Fermat’s principle: if material is homogeneous, light travels on a straight line
- NOT so for inhomogeneous medium
**Perspective geometry**

- Define image plane and centre of projection
- All points that are on the same line from a centre of projection cover each other
- Projection maps 3D to 2D

- Image plane can be before or behind the centre of projection
- Mathematical modeling relatively simple

![Diagram showing perspective geometry with an object, image plane, center of projection, and eye point.](image-url)
Projective geometry

- Geometry:
  - Elements of set S
  - Transformation group T: one binary operation satisfying closure, identity, inverse and associativity

- In perspective geometry, transformations are linear, i.e. in matrix form

- For n-dimensional perspective geometry:
  - S (points): \((x_0, x_1, \ldots, x_n)\) except the centre of projection \((0,0,\ldots,0)\)
    - De facto, lines passing through the origin
    - By convention, the origin is centre of projection
  - T: Invertible \((n+1,n+1)\) matrices

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![Projective geometry diagram](image)
Projective geometry

• Properties of projective geometry:
  – Straight lines are mapped into straight lines
  – Incidence relation is preserved
  – Cross ratio is preserved
  – Images of parallel lines intersect at a vanishing point

• Fundamental theorem:
  – $n+2$ independent points are enough to determine a unique projective transformation in $n$-dimensional projective geometry

• Consequence:
  – 4 chromaticity points are enough to determine the transformation from one colour system to another one
Projective geometry

- In 3D space, we will use 3D projective geometry
- Transformations are 4x4 invertible matrices
- Thus, transforming \((x,y,z,t)\) into \((x',y',z',t')\):

\[
\begin{bmatrix}
  x' \\
  y' \\
  z' \\
  t'
\end{bmatrix} = \begin{bmatrix}
  m_{11} & m_{12} & m_{13} & m_{14} \\
  m_{21} & m_{22} & m_{23} & m_{24} \\
  m_{31} & m_{32} & m_{33} & m_{34} \\
  m_{41} & m_{42} & m_{43} & m_{44}
\end{bmatrix} \begin{bmatrix}
  x \\
  y \\
  z \\
  t
\end{bmatrix} = M \begin{bmatrix}
  x \\
  y \\
  z \\
  t
\end{bmatrix}
\]

- The inverse is easy: if \((x',y',z',t')\) can be rewritten as \((x'',y'',z'',1)\) by putting \(x''=x'/t', y''=x'/t', z''=z'/t'\), and

\[
\begin{align*}
  x'' &= \frac{m_{11}x + m_{12}y + m_{13}z + m_{14}t}{m_{41}x + m_{42}y + m_{43}z + m_{44}t} \\
  y'' &= \frac{m_{21}x + m_{22}y + m_{23}z + m_{24}t}{m_{41}x + m_{42}y + m_{43}z + m_{44}t} \\
  z'' &= \frac{m_{31}x + m_{32}y + m_{33}z + m_{34}t}{m_{41}x + m_{42}y + m_{43}z + m_{44}t}
\end{align*}
\]

these are called the projective transformations.
Geometrical theory of optical imaging

- In an ideal system, a perfectly focused image would form on the image plane.
- Sharp image point: all rays that originate from a point in object space can be refracted so that they convey to a single point in image space.
- Sharp image: sharp at all image points.
- This is not the case in typical photographic images.
A typical optical system

- Imaging systems are complex:
Optical Image Formation

- Images are formed by focusing light onto a sensor
- On real life, not all the light available can be collected onto the sensor
- Because camera systems collect only a part of the wavefront, diffraction will limit the optical imaging system
- If sensors are large enough WRT wavelength, diffraction can be neglected, and geometrical optics can be used

- In geometric optics, the following things are considered valid:
  - Fermat’s principle
  - Snell’s law
  - Eikonal equation
  - Ray equation
- Consider a point light source: rays emanating from it will diverge
- We can call the source a focus of a bundle of rays
- If a ray bundle with some optical system can be made to converge to a single point we call this point a focus point.
Optical Image Formation

- **Stigmatic** (sharp) optical system: A ray bundle generated at a point \( P_0 \) can be made entirely converge to another point \( P_1 \).
- \( P_0, P_1 \) *conjugate points*: reversing their roles a perfect image of \( P_1 \) would be created at \( P_0 \).
- If the rays instead converge to a small area, blur occurs and the image is not perfect.
- An optical system may allow points nearby \( P_0 \) to be stigmatically imaged to points that are nearby \( P_1 \).
- In *Ideal optical system*, the region of points that are stigmatically imaged is called *object space*.
- The region of points into which object space is stigmatically imaged is called *image space*.
- Both these spaces are 3D.
- *Perfect image*: a curve in object space maps to an identical curve in image space.
Absolute instruments

• An optical system that is stigmatic and perfect is called an absolute instrument.

• For absolute instruments, following applies:
  – Maxwell’s theorem for absolute instruments: the optical length of any curve in object space equals the optical length of its image.

  – Charatheodory’s theorem: the mapping between object and image space of an absolute instrument is either a projective transformation, an inversion, or a combination of both

• Restrictions on absolute instruments are too heavy

• In most practical imaging systems, the image space is a part of a plane or of a surface and is called the image plane.
Imaging Geometry: first-order optics

- Assumption: the optical imaging system is such that all rays only make a small angle $\Phi$ WRT a reference axis.
  - Such rays are called *paraxial*.
  - In such systems, sinus and cosinus can be approximated:
    - $\sin(\Phi) \approx \Phi$
    - $\cos(\Phi) \approx 1$

- *Linear optics*
  - Additionally, all optical elements are arranged along a reference axis, called *optical axis*.
  - And all elements are rotationally symmetric WRT optical axis.
  - This is called *Gaussian*, or *paraxial*, or *first-order optics*.
  - Imaging can be here approximated through projective transformations.

- Object point $P = (p_x, p_y, p_z)^T$ maps to $P' = (p'_x, p'_y, p'_z)^T$ through
  \[
  \begin{align*}
  p'_x &= \frac{m_{11}p_x + m_{12}p_y + m_{13}p_z + m_{14}}{m_{41}p_x + m_{42}p_y + m_{43}p_z + m_{44}}, \\
  p'_y &= \frac{m_{21}p_x + m_{22}p_y + m_{23}p_z + m_{24}}{m_{41}p_x + m_{42}p_y + m_{43}p_z + m_{44}}, \\
  p'_z &= \frac{m_{31}p_x + m_{32}p_y + m_{33}p_z + m_{34}}{m_{41}p_x + m_{42}p_y + m_{43}p_z + m_{44}}.
  \end{align*}
  \]

In homogenous coordinates and through symmetry we can write:

\[
\begin{bmatrix}
  p'_x \\
  p'_y \\
  p'_z \\
  p'_w
\end{bmatrix} =
\begin{bmatrix}
  f & 0 & 0 & 0 \\
  0 & f & 0 & 0 \\
  0 & 0 & z'_0 & f f' - z_0 z'_0 \\
  0 & 0 & 1 & -z_0
\end{bmatrix}
\begin{bmatrix}
  p_x \\
  p_y \\
  p_z \\
  p_w
\end{bmatrix}
\]

$z_0, z'_0$: focal points
$f, f'$: focal lengths

- The 3D position of the transformed point is found by dividing by the homogeneous coordinate:
  $P' = (p'_x/p'_w, p'_y/p'_w, p'_z/p'_w)$
Imaging geometry

- The optical system sits somewhere between P and P' and is centered around the z axis.
- Right handed coords pointed as z (optical axis).
- y points up.
- The x = 0-plane is called the **meridional plane**.
- Rays lying in this plane are called **meridional rays**.
- All other rays are called **skew rays**.
- Meridional rays passing through an optical system stay in the meridional plane.

![Diagram showing imaging geometry](image-url)
Imaging geometry

- For an isotropic system (rotationally symmetric), one can drop the x coordinate.
- The perspective becomes Newton’s equation:
  \[ p'_y = \frac{fp_y}{z - z_0} \]
  and the \( z \) is given by
  \[ p'_z - z_0' = \frac{ff'}{z - z_0} \]
- This equation is the perspective transformation for a pinhole camera.
- Pinhole camera: small hole in a surface separating object from image space.

\( P_0 = (p_x, p_z) \)
\( f' \)
\( f \)
\( z_0 \)
\( P_1 = (p'_x, p'_z) \)

\( N = \) Object nodal point
\( N' = \) Image nodal point
\( F = \) Object focal point
\( F' = \) Image focal point
\( H = \) Object principal point
\( H' = \) Image principal point
Imaging geometry

• Several points are important:
  – Object focal point (front focal point) $F=(0,0,z_0)^T$
  – Image focal point (back focal point) $F'=(0,0,z'_0)^T$
  – Object principal point (front principal point) $H=(0,0,z_0+f)^T$.
    The plane // to xy passing through $H$ is called object principal plane
  – Objects on the principal plane are imaged with a magnification of 1.
  – Image principal point $H'=(0,0,z'_0+f')^T$
  – Object nodal point $N=(0,0,z_0-f')^T$
    a ray passing through $N$ at angle $\theta$ with the optical axis will pass through $N'$ at the same angle
  – Image nodal point $N'=(0,0,z'_0-f)$
Imaging geometry

- In a real system, the radius of the lens is limited.
- Thus only a portion of the light emitted by the light source will reach the image.
- The smallest diameter through which light passes is determined by the lens or an adjustable diaphragm (aperture stop).
- The element limiting the angular extent of the object to be imaged is called field stop.
- Field of view.
- Entrance pupil: aperture seen by a point on optical axis and on object
  - Size determined by aperture + lenses between obj and aperture stop
- Exit pupil: aperture seen from the image plane through any lenses located between aperture and image plane.
- Ratio entrance/exit pupil: pupil magnification
- Chief ray: start from any off-axis point on the object and going through center of aperture stop.
- Marginal ray: starts from on axis point on object and passes through entrance pupil.
Imaging radiometry

- A camera is: optical system + sensor
- Sensor measures image irradiance $E_e$ resulting from scene radiance $L_e$ incident through optical system
- We now want to study their relationship

- Following assumptions are made:
  - Object distance large with respect to focal length
  - $E_e$ proportional to entrance pupil
  - $E_e$ inversionally proportional to square of focal length $f^2$. This because lateral magnification is proportional to focal length: the longer the focal length, the larger the area covered by the image
**Imaging radiometry**

- Differential area $dA$, off-axis in the object plane, projecting to a corresponding differential area $dA'$ on image plane.
- Between these areas there is the optical system.
- Chief ray from $dA$ makes angle $\theta$ with optical axis.
- $s$: distance $dA$ entrance pupil
- $h$: distance from optical axis
- $d$: radius entrance pupil
- $d\Psi$: diff. area on entrance pupil at distance $r$ from optical axis.
Imaging radiometry

- We want to integrate over entrance pupil, i.e. sum $d\Psi$
- Vector $\mathbf{v}$ from $dA$ to $d\Psi$:

$$
\mathbf{v} = \begin{bmatrix}
  r \cos(\Psi) \\
  r \sin(\Psi) - h \\
  s
\end{bmatrix}
$$

- $\mathbf{v}$ makes an angle $\alpha$ with optical axis, computable from

$$
\cos(\alpha) = \frac{s}{||\mathbf{v}||}
$$
Imaging radiometry

- If $dA$ is lambertian then the flux incident into $dA'$ is

\[
d\Phi_0 = L_e \int_{r=0}^{d} \int_{\Psi=0}^{2\pi} \frac{r d\Psi dr \frac{s}{\|v\|}}{\|v\|^2} dA \frac{s}{\|v\|}
\]

\[
= L_e \int_{r=0}^{d} \int_{\Psi=0}^{2\pi} \frac{r s^2 d\Psi dr}{(r^2 \cos^2(\Psi) + (r \sin(\Psi) - h)^2 + s^2)^2} dA
\]

\[
= L_e dA \int_{r=0}^{d} \frac{2\pi (s^2 + h^2 + r^2) r s^2 dr}{(s^2 + h^2 + r^2)^3/2}
\]

\[
= \frac{\pi}{2} L_e dA \left(1 - \frac{s^2 + h^2 - d^2}{(s^2 + h^2 + d^2)^3/2} \right).
\]

- Similarly for quantities at the exit pupil (indicated with ‘)
Imaging radiometry

- If the optical system has no light losses, flux at entrance and exit pupils are the same:
  \[ E'_e = \frac{d\Phi_0}{dA'} \]
  \[ = \frac{\pi L_e}{2} \frac{dA}{dA'} \left( 1 - \frac{s^2 + h^2 - d^2}{\left( s^2 + h^2 + d^2 \right)^2 - 4h^2d^2} \right)^{1/2} \]

- This is equivalent to
  \[ = \frac{\pi L'_e}{2} \left( 1 - \frac{s^2 + h'^2 - d'^2}{\left( s'^2 + h'^2 + d'^2 \right)^2 - 4h'^2d'^2} \right)^{1/2} \]

- Similarly for quantities at the exit pupil (indicated with ')
  \[ d\Phi_1 = \frac{\pi}{2} L'_e dA' \left( 1 - \frac{s'^2 + h'^2 - d'^2}{\left( s'^2 + h'^2 + d'^2 \right)^2 - 4h'^2d'^2} \right)^{1/2} \]
Imaging radiometry

• Call:
  – $n$ refraction index at object plane
  – $n'$ refraction index at image plane

• Then:

\[
E'_e = \frac{\pi}{2} L'_e \left( \frac{n'}{n} \right)^2 \left( 1 - \frac{s'^2 + h'^2 - d'^2}{(s'^2 + h'^2 + d'^2)^2 - 4h'^2d'^2} \right)^{1/2}
\]

\[
= \frac{\pi}{2} L'_e \left( \frac{n'}{n} \right)^2 G,
\]

where

\[
G = 1 - \frac{s'^2 + h'^2 - d'^2}{(s'^2 + h'^2 + d'^2)^2 - 4h'^2d'^2} \right)^{1/2}
\]

*Image Irradiance Equation*

• IIE is general, but hard to compute
• It can be simplified for certain cases: for example for on-axis imaging, as well as for off-axis imaging
  – Object distance much larger than entrance pupil
On axis image irradiance

• When object of interest is on optical axis, then $h=h'=0$. The equation simplifies to:

$$E'_e = \pi L_e \left( \frac{n'}{n} \right)^2 \left( \frac{d'^2}{s'^2 + d'^2} \right)$$

Consider the cone spanned by the exit pupil as the base and the on-axis point on the image plane as the apex:

• then the sine of the half-angle $\beta$ of this cone is given by:

$$\sin(\beta) = \frac{d'^2}{\sqrt{s'^2 + d'^2}}$$

substituting:

$$E'_e = \frac{\pi L_e}{n^2} \left( n' \sin(\beta) \right)^2$$

• $n' \sin(\beta)$ is called numerical aperture
• $E'_e$ is proportional to numerical aperture: the larger the aperture, the lighter the image (speed of system)
• A related measure is the relative aperture $F$ (f-number):

$$F = \frac{1}{2n' \sin(\beta)}$$

• If image point at infinity, then one can assume distance between image plane and exit pupil $s' = \text{image focal length } f'$
• And $\beta \approx \tan^{-1}(d'/f')$ so relative aperture becomes

$$F_\infty \approx \frac{1}{2n' \sin(\tan^{-1}(d'/f'))}$$

$$\approx \frac{1}{n' \frac{f'}{2d'}}.$$
On axis image irradiance

- Using pupil magnification \( m_d = d/d' \) we can rewrite as
  \[
  F_\infty \approx \frac{1}{m_p n} \frac{f}{2d}
  \]
  if object and image plane are in air, then refraction index is 1

- If magnification factor is close to 1, then relative aperture for object at infinity can be approximated:
  \[
  F_\infty = \frac{f}{D}
  \]
  where \( D = \) diameter of entrance pupil

- An alternative notation for the f-number is \( f/N \), where \( N \) is replaced by \( f/D \)

- So, for a lens of focal length of 50mm and aperture of 8.9mm, the f-number is written as \( f/5.6 \).

- Image irradiance can be written as:
  \[
  E'_e = \frac{\pi D^2 L_e}{4} \left( \frac{m_p}{f} \right)^2
  \]
  notice: \( \pi D^2/4 = \) area of entrance pupil
Off axis image irradiance

- For objects not on optical axis we can assume distance to entrance pupil much bigger than entrance pupil radius ($s \gg d$): irradiance is approximated as:

\[ E'_e \approx \pi L_e \frac{s^2 d^2}{(s^2 + d^2 + h^2)^2} \frac{dA}{dA'} \]

\[ \approx \pi L_e \frac{s^2 d^2}{(s^2 + h^2)^2} \frac{dA}{dA'} \]

look at picture: cosine of off axis angle $\theta$ is

\[ \cos(\theta) = \frac{s}{\sqrt{s^2 + h^2}} \]

thus image irradiance becomes

\[ E'_e \approx \pi L_e \cos^4(\theta) \left(\frac{d}{s}\right)^2 \frac{dA}{dA'} \]

now dA/dA' is related to lateral magnification of the lens $m$ through

\[ m = \sqrt{\frac{dA}{dA'}} \]

- So:

\[ E'_e \approx \pi L_e \cos^4(\theta) \left(\frac{d}{s}\right)^2 m^2 \]

lateral magnification satisfies

\[ \frac{m}{m-1} = \frac{f'}{s} \]

thus

\[ E'_e \approx \pi L_e \cos^4(\theta) \left(\frac{d}{(m-1) f'}\right)^2 \]

or in terms of t-number

\[ E'_e \approx \frac{\pi L_e}{4 F^2 n^2 (m-1)^2 m_p^2} \cos^4(\Theta) \]

for $m=2$, $m_p=1$, refraction at image is 1, so the falloff is

\[ E'_e \approx \frac{\pi L_e}{4 F^2} \cos^4(\Theta) \approx \frac{\pi L_e}{4} \frac{d^2}{f^2} \cos^4(\Theta) \]
Off axis image irradiance

- The consequence? Light falloff!
- Modern lenses tend to perform better than $\cos^4$
Vignetting

• For simple opt.sys. as in picture the dimension of lenses impose an aperture
• The cross-section of aperture depends on which point in object plane is used
• Further off axis=smaller cross-section

• So, less light arrives to image space, so additional fall-off called *vignetting*
• Amount depends on distance to optical axis
• We introduce
  – spatial dependency on points in the object plane \((x,y)\) and corresponding points on the image plane \((x',y')\),
  – attenuation factor \(V(x',y')\) that takes vignetting into consideration

Irradiance becomes:

\[
E'_e(x',y') = \frac{\pi}{2} I'_e(x',y') \cdot V(x',y') \left(\frac{n'}{n}\right)^2 G
\]
Glare
• Optical systems have many imperfections not taken into account by the irradiance equation.

• Lens barrel and aperture blades might scatter light, so some light will be smeared all over the image plane: veiling glare or lens flare.

• Frequent by looking at light sources.

• Others might result from reflections inside the lens.

• Modeling glare for the irradiance can be done by adding a glare function \( g(x', y') \)

\[
E'_e(x', y') = \frac{\pi}{2} L'_e(x', y') T V(x', y') \left( \frac{n'}{n} \right)^2 G + g(x', y')
\]

the more components a lens has, the more prone it is to glare.

• Especially true in zoom lenses.
End

• Thank you for listening!