Fundamentals of Imaging
Geometrical optics

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This slide pack

- In this part, we will introduce geometrical optics:
  - Principles of geometrical optics
  - Fermat’s principle
  - Perspective-projective geometry
  - Optical systems
    - Optical image formation
  - Absolute instruments
  - Imaging geometry
  - Imaging radiometry
  - On-axis and off-axis irradiance
  - Effects: Vignetting, glare

Image capture

- Imaging:
  - mapping of some characteristics of the real world (object space)
  - into another representation of this space (image space)
- In general, a capturing system will be composed of several components
  - Components are optimized to convey light to the sensing device
  - Several variables are available here, and they affect the quality of the system
- Despite knowing that light is generated by quantum mechanics
- In general one would use the geometric (optical) representation of light for this
- Main assumption:
  - Light can be treated as rays, because its wavelength is less than 1 micron
  - Neglectable with respect to distances travelled
  - Characteristics can be studied geometrically
  - Whenever light has to be treated as waves, one has to do it explicitly

The basis of geometrical optics

- An arbitrary complex time function of the electromagnetic field can be decomposed into Fourier components of time harmonics
- Let us take a general time harmonic field[1]:

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0(\mathbf{r})e^{-i\omega t},$$

$$\mathbf{H}(\mathbf{r}, t) = \mathbf{H}_0(\mathbf{r})e^{-i\omega t}$$

  in regions free of currents and charges, $\mathbf{E}_0$ and $\mathbf{H}_0$ will satisfy time-free Maxwell equations.
- Define $k_0=2\pi/\lambda_0$, where is the wavelength in vacuum.
- Away from the source, the fields can be represented as general fields

$$\mathbf{E}_0(\mathbf{r}) = \mathbf{e}(\mathbf{r})e^{-ik_0\psi(\mathbf{r})},$$

$$\mathbf{H}_0(\mathbf{r}) = \mathbf{h}(\mathbf{r})e^{-ik_0\psi(\mathbf{r})}$$

- Assuming that $\lambda_0 \rightarrow 0$, and that terms containing $1/k_0$ can be neglected, from Maxwell's equation one can derive

$$\nabla\psi \cdot \nabla\psi = \left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2 = n^2(x, y, z)$$

  eikonal equation
  n: index of refraction
  ψ: eikonal function

  nabla operator $\vec{\nabla} = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$
- Where ψ constant phases are constant (geometrical wavefronts)
- Energy of the electromagn. wave propagates with velocity $v=c/n$ in the surface normal to the wavefronts
- Thus light rays are orthogonal to the geometrical wavefronts

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(1) In this chapter, bold variables will represent vectors

The basis of geometrical optics

- Let
  - $\mathbf{r}(s)$ position vector of a point on a light ray,
  - s arc length of ray,
  - Then $d\mathbf{r}/ds$ is a unit vector pointing to the direction of the light ray
- One can then rewrite the eikonal equation as

$$n\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \nabla\psi$$

- Because the distance between two neighbouring wavefronts $\mathrm{d}\psi$ can be expressed as

$$\mathrm{d}\psi = \mathrm{d}\mathbf{r} \cdot \nabla\psi = n\mathrm{d}s,$$

the integral $\int_{P1}^{P2} n\mathrm{d}s$

taken on a curve along the path from P1 to P2 is called the optical path length between the points

- In most cases, the light ray travels along the path of shortest optical length
- However, this is not always true:
  - Light rays travel along the path that have zero derivative with respect to time or with respect to the optical path length (Fermat's principle)
- Because the light ray is gradient of a scalar field, then if the ray vector is operated by a curl operator, the result is zero
- This proves Snell's law: incident ray, refracted day and surface normal are all in the same plane

Fermat’s principle

- Eikonal equation describes geometrical optics
- Alternatively, one can use *Fermat’s principle*: light follows a ray such that optical path length is an extremum
- Optical path length: $\int_a^b n\, ds$

  ds: arc length

  n refraction index

  a,b: start and end of path
- Minimizing this integral through variation calculus results in the *ray equation*

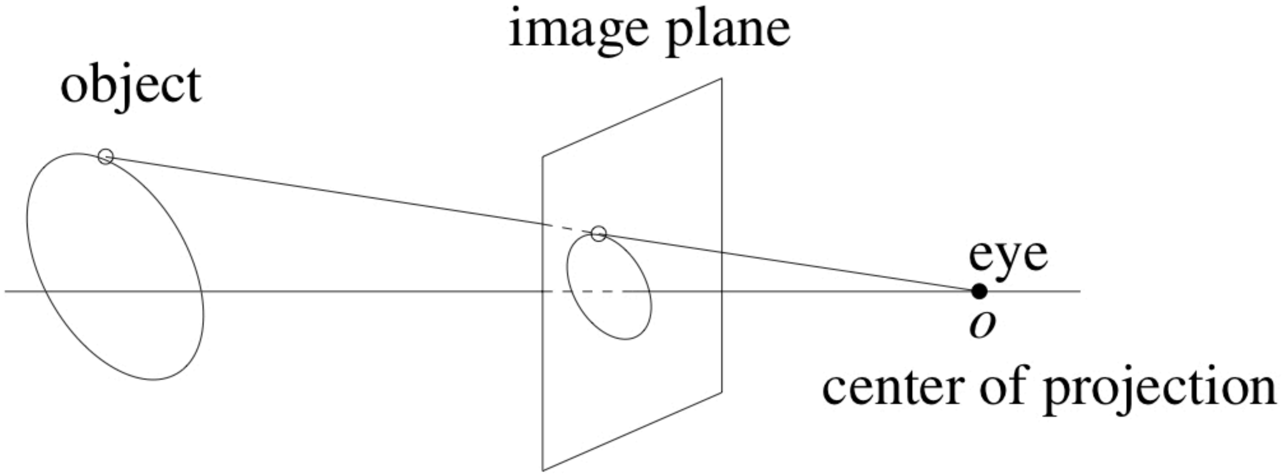
$$\frac{d}{d\mathbf{l}}\left(n(\mathbf{r})\frac{d\mathbf{r}}{d\mathbf{l}}\right) = \nabla n(\mathbf{r})$$

- Meaning:
  - at every point of the medium, tangent and normal of a ray form a plane, called osculating plane
  - The gradient of the refracting index must lie in this plane
- Valid for inhomogeneous isotropic media which are stationary over time

- A consequence of Fermat’s principle: if material is homogeneous, light travels on a straight line
- NOT so for inhomogeneous medium

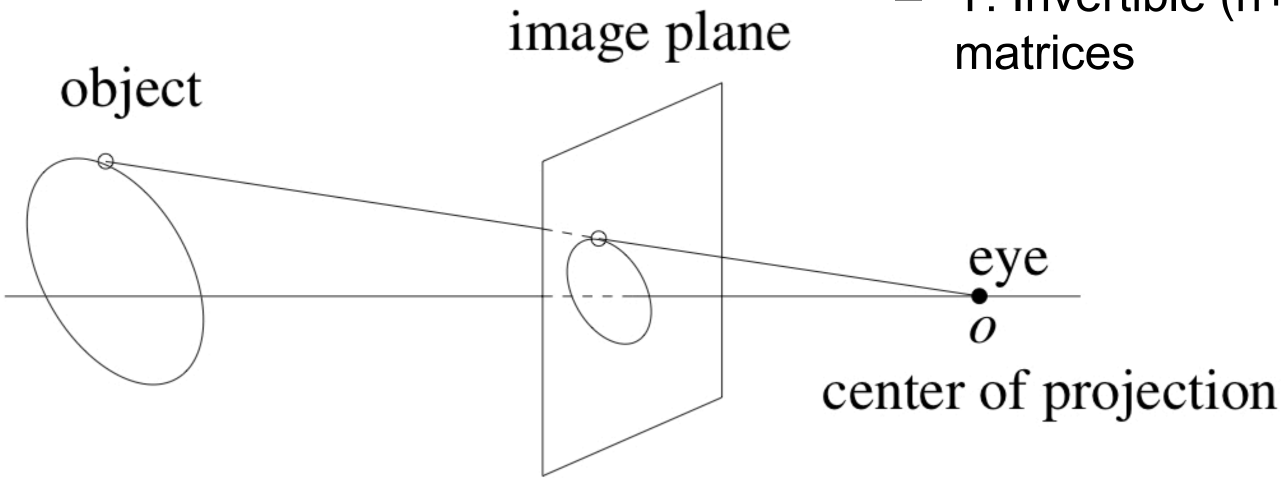
Perspective geometry

- Define image plane and centre of projection
- All points that are on the same line from a centre of projection cover each other
- Projection maps 3D to 2D
- Image plane can be before or behind the centre of projection
- Mathematical modeling relatively simple



Projective geometry

- Geometry:
  - Elements of set S
  - Transformation group T: one binary operation satisfying closure, identity, inverse and associativity
- In perspective geometry, transformations are linear, i.e. in matrix form



- For n-dimensional perspective geometry:
  - S (points): $(x_0,x_1,\ldots,x_n)$ except the centre of projection $(0,0,\ldots,0)$
    - De facto, lines passing through the origin
    - By convention, the origin is centre of projection
  - T: Invertible (n+1,n+1) matrices

Projective geometry

- Properties of projective geometry:
  - Straight lines are mapped into straight lines
  - Incidence relation is preserved
  - Cross ratio is preserved
  - Images of parallel lines intersect at a vanishing point
- Fundamental theorem:
  - n+2 independent points are enough to determine a unique projective transformation in n-dimensional projective geometry
- Consequence:
  - 4 chromaticity points are enough to determine the transformation from one colour system to another one

Projective geometry

- In 3D space, we will use 3D projective geometry
- Transformations are 4x4 invertible matrices
- Thus, transforming (x,y,z,t) into (x’,y’,z’,t’):

$$\begin{bmatrix} x' \\ y' \\ z' \\ t' \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = M \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$$

- The inverse is easy:
  if (x’,y’,z’,t’) can be rewritten as (x”,y”,z”,1) by putting x”=x’/t’, y”=x’/t’, z”=z’/t’, and

$$x'' = \frac{m_{11}x + m_{12}y + m_{13}z + m_{14}t}{m_{41}x + m_{42}y + m_{43}z + m_{44}t}$$

$$y'' = \frac{m_{21}x + m_{22}y + m_{23}z + m_{24}t}{m_{41}x + m_{42}y + m_{43}z + m_{44}t}$$

$$z'' = \frac{m_{31}x + m_{32}y + m_{33}z + m_{34}t}{m_{41}x + m_{42}y + m_{43}z + m_{44}t}$$

these are called the projective transformations

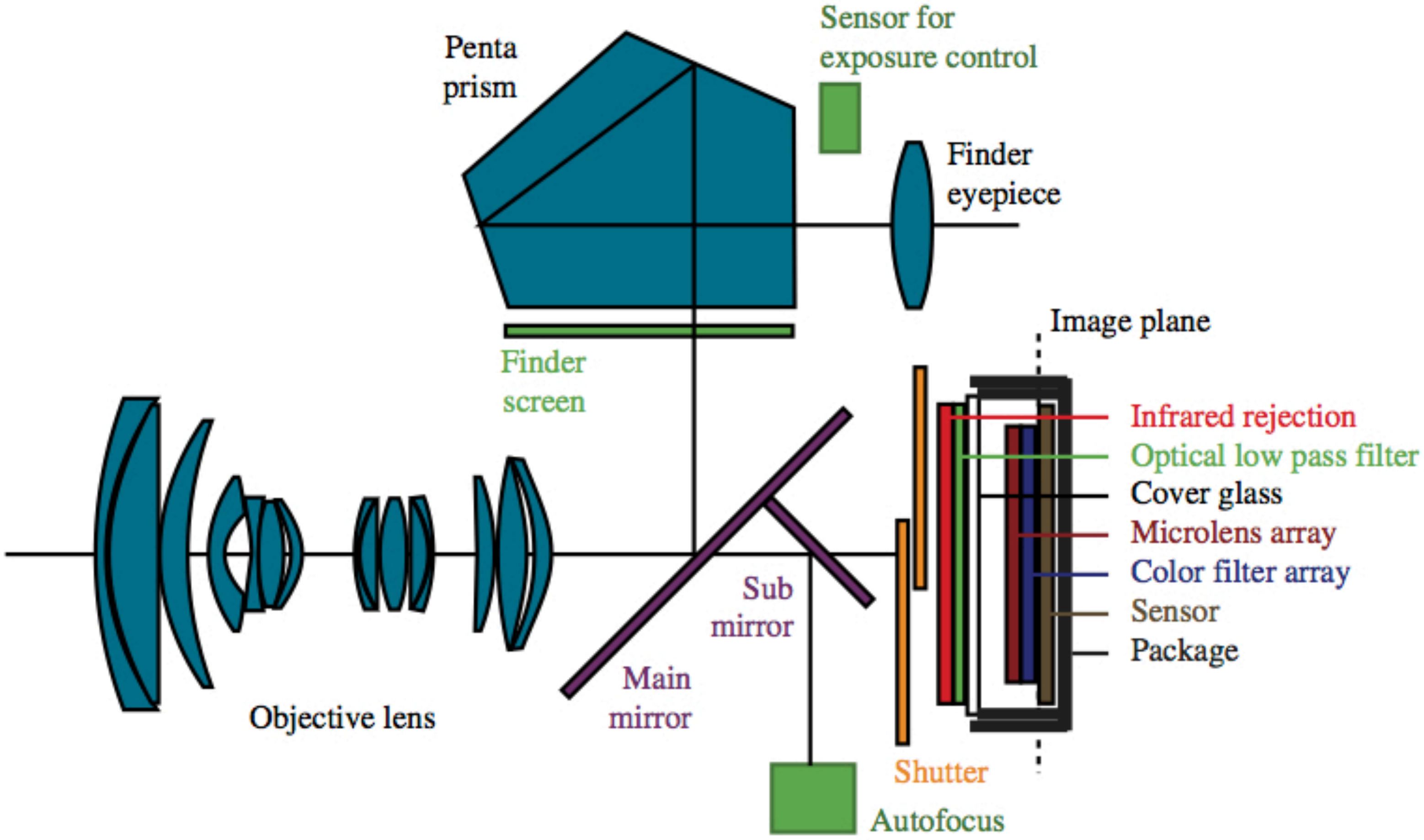
Geometrical theory of optical imaging

- In an ideal system, a perfectly focused image would form on the image plane
- Sharp image point: all rays that originate from a point in object space can be refracted so that they convey to a single point in image space
- Sharp image: sharp at all image points

- This is not the case in typical photographic images

A typical optical system

- Imaging systems are complex:



Optical Image Formation

- Images are formed by focusing light onto a sensor
- On real life, not all the light available can be collected onto the sensor
- Because camera systems collect only a part of the wavefront, diffraction will limit the optical imaging system
- If sensors are large enough WRT wavelength, diffraction can be neglected, and geometrical optics can be used
- In geometric optics, the following things are considered valid:
  - Fermat's principle
  - Snell's law
  - Eikonal equation
  - Ray equation
- Consider a point light source: rays emanating from it will diverge
- We can call the source a *focus* of a bundle of rays
- If a ray bundle with some optical system can be made to converge to a single point we call this point a *focus point*.

Optical Image Formation

- *Stigmatic* (sharp) optical system: A ray bundle generated at a point P_0 can be made entirely converge to another point P_1.
- P_0, P_1 *conjugate points*: reversing their roles a perfect image of P_1 would be created at P_0.
- If the rays instead converge to a small area, blur occurs and the image is not perfect
- An optical system may allow points nearby P_0 to be stigmatically imaged to points that are nearby P_1.
- In *Ideal optical system*, the region of points that are stigmatically imaged is called *object space*
- The region of points into which object space is stigmatically imaged is called *image space*.
- Both these spaces are 3D
- *Perfect image*: a curve in object space maps to an identical curve in image space.

Absolute instruments

- An optical system that is stigmatic and perfect is called an *absolute instrument*.
- For absolute instruments, following applies:
  - Maxwell's theorem for absolute instruments:
    the optical length of any curve in object space equals the optical length of its image.
  - Charatheodory's theorem:
    the mapping between object and image space of an absolute instrument is either a projective transformation, an inversion, or a combination of both
- Restrictions on absolute instruments are too heavy
- In most practical imaging systems, the image space is a part of a plane or of a surface and is called the *image plane*.

Imaging Geometry: first-order optics

- Assumption: the optical imaging system is such that all rays only make a small angle Φ WRT a reference axis
- Such rays are called *paraxial*
- In such systems, sinus and cosinus can be approximated:
  - $\sin(\Phi) \approx \Phi$
  - $\cos(\Phi) \approx 1$
- *Linear optics*
- Additionally, all optical elements are arranged along a reference axis, called *optical axis*.
- And all elements are rotationally symmetric WRT optical axis
- This is called *Gaussian*, or *paraxial*, or *first-order optics*
- Imaging can be here approximated through projective transformations

- Object point $P=(p_x,p_y,p_z)^T$ maps to $P'=(p'_x,p'_y,p'_z)^T$ through

$$p'_x = \frac{m_{11}p_x + m_{12}p_y + m_{13}p_z + m_{14}}{m_{41}p_x + m_{42}p_y + m_{43}p_z + m_{44}},$$
$$p'_y = \frac{m_{21}p_x + m_{22}p_y + m_{23}p_z + m_{24}}{m_{41}p_x + m_{42}p_y + m_{43}p_z + m_{44}},$$
$$p'_z = \frac{m_{31}p_x + m_{32}p_y + m_{33}p_z + m_{34}}{m_{41}p_x + m_{42}p_y + m_{43}p_z + m_{44}}.$$

in homogenous coordinates and through symmetry we can write

$$\begin{bmatrix} p'_x \\ p'_y \\ p'_z \\ p'_w \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & z'_0 & ff' - z_0 z'_0 \\ 0 & 0 & 1 & -z_0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

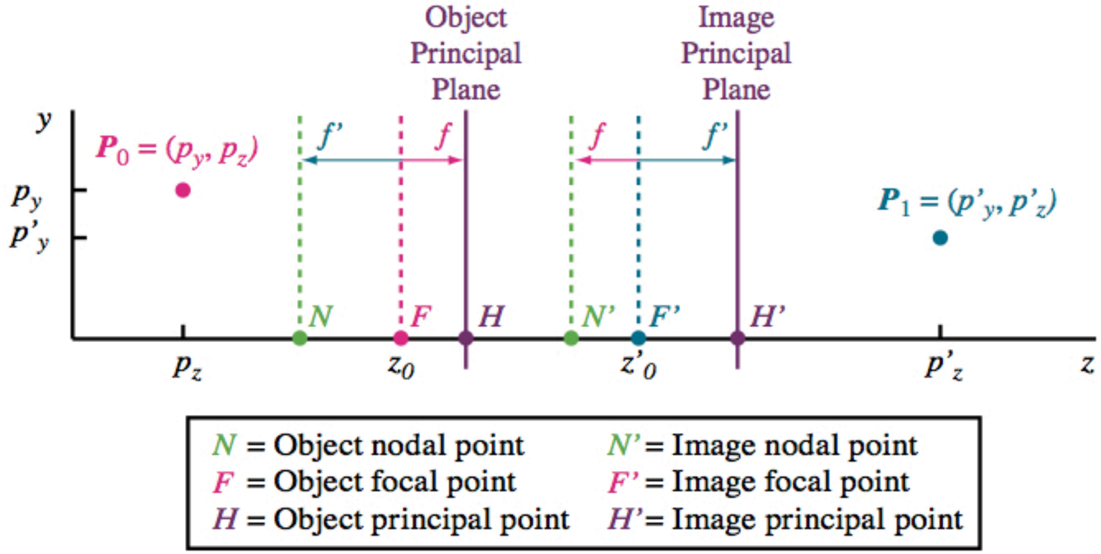
z_0,z'_0:focal points
f,f': focal lengths

- The 3D position of the transformed point is found by dividing by the homogeneous coordinate:
  $P'=(p'_x/p'_w, p'_y/p'_w, p'_z/p'_w)$

Imaging geometry

- The optical system sits somewhere between P and P’ and is centered around the z axis
- Right handed coords pointed as z (optical axis)
- y points up

- The $x = 0$-plane is called *meridional plane*
- Rays lying in this plane are called *meridional rays*.
- All other rays called *skew rays*.
- Meridional rays passing through an optical system stay in the meridional plane.



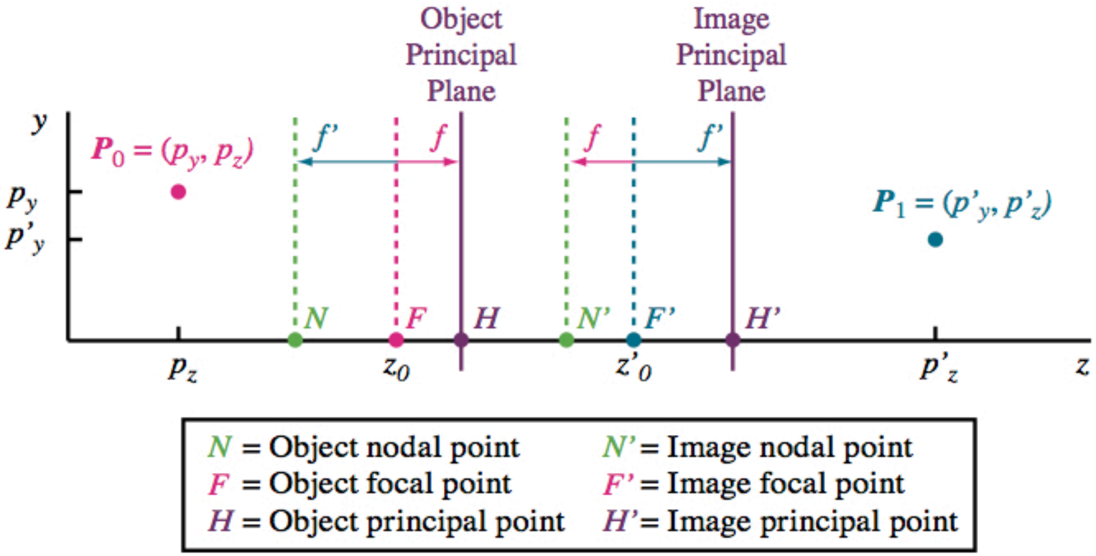
Imaging geometry

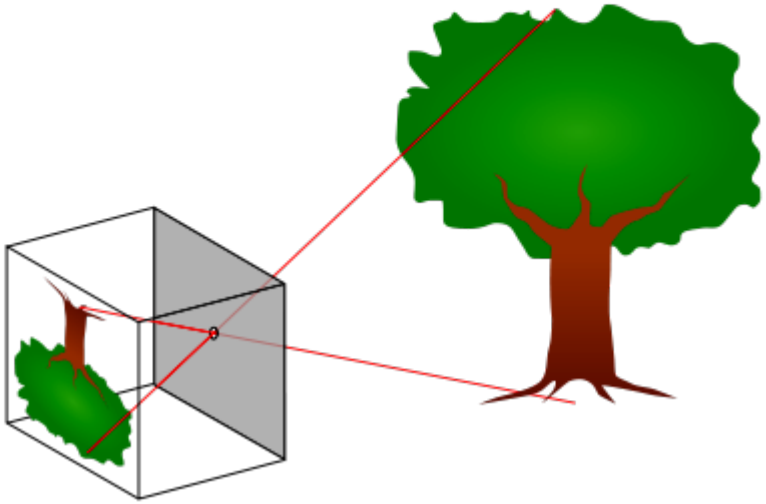
- For an isotropic system (rotationally symmetric), one can drop the x coordinate
- The perspective becomes *Newton's equation*

$$p'_y = \frac{f p_y}{z - z_0}$$

and the z is given by $p'_z - z'_0 = \frac{f f'}{z - z_0}$

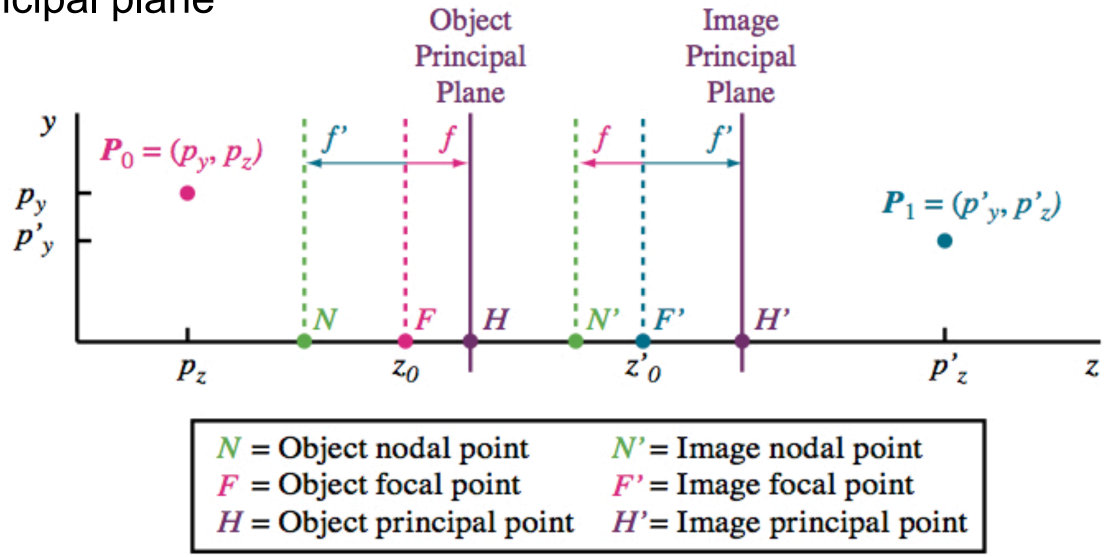
- This equation is the perspective transformation for a pinhole camera
- Pinhole camera: small hole in a surface separating object from image space





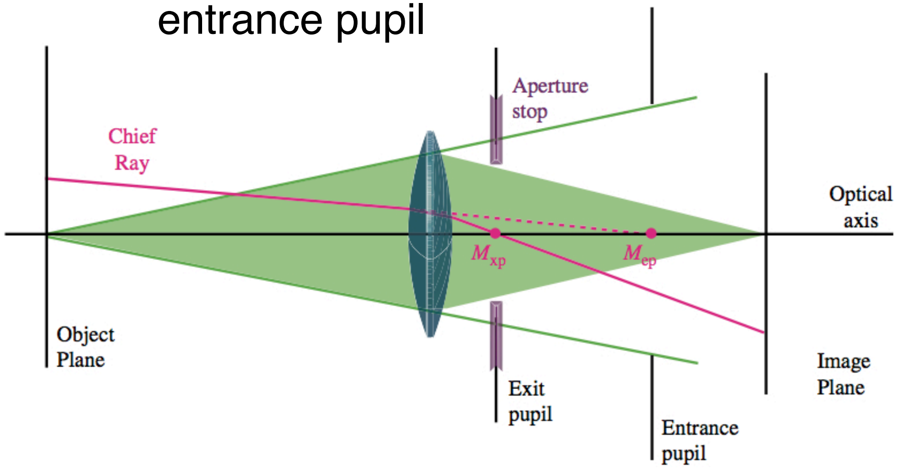
Imaging geometry

- Several points are important:
  - Object focal point (front focal point) $F=(0,0,z_0)^T$
  - Image focal point (back focal point) $F'=(0,0,z'_0)^T$
  - Object principal point (front principal point) $H=(0,0,z_0+f)^T$. The plane // to xy passing through H is called object principal plane
- Objects on the principal plane are imaged with a magnification of 1.
- Image principal point $H'=(0,0,z'_0+f')^T$
- Object nodal point $N=(0,0,z_0-f')^T$ a ray passing through N at angle θ with the optical axis will pass through N' at the same angle
- Image nodal point $N'=(0,0,z'_0-f)$



Imaging geometry

- In a real system, the radius of the lens is limited
- Thus only a portion of the light emitted by the light source will reach the image
- The smallest diameter through which light passes is determined by the lens or an adjustable diaphragm (*aperture stop*)
- The element limiting the angular extent of the object to be imaged is called *field stop*.
- *Field of view*.
- *Entrance pupil*: aperture seen by a point on optical axis and on object
  - Size determined by aperture + lenses between obj and aperture stop
- *Exit pupil*: aperture seen from the image plane through any lenses located between aperture and image plane
- Ratio entrance/exit pupil: *pupil magnification*
- *Chief ray*: start from any off-axis point on the object and going through center of aperture stop
- *Marginal ray*: starts from on axis point on object and passes through entrance pupil



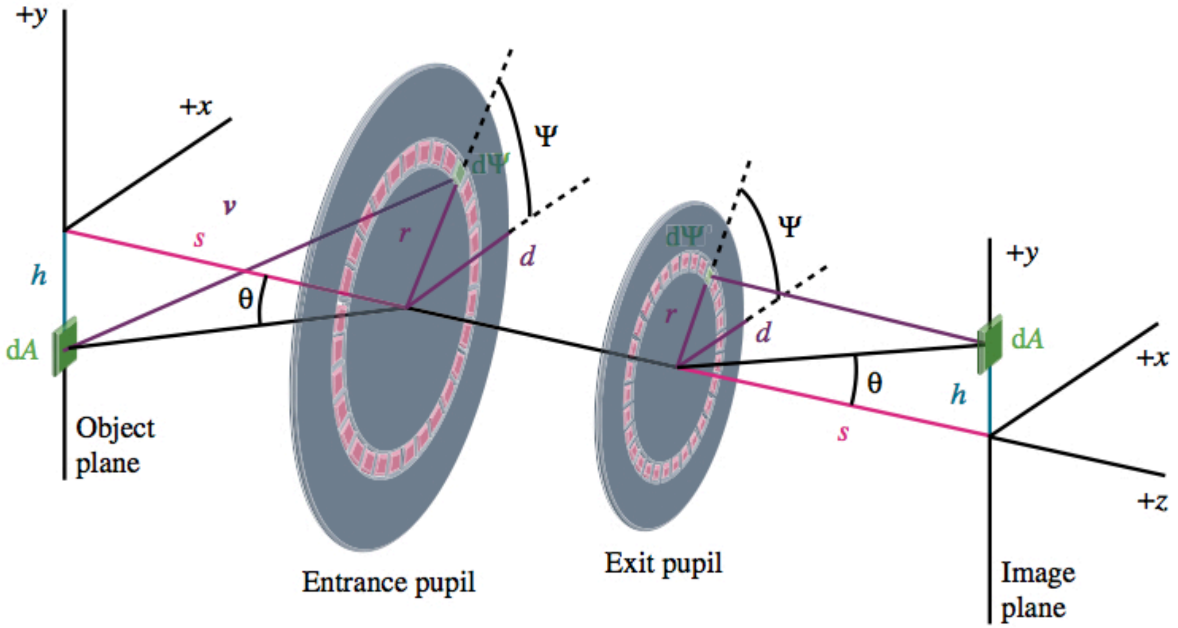
Imaging radiometry

- A camera is:
  optical system + sensor
- Sensor measures image irradiance E_e resulting from scene radiance L_e incident through optical system
- We now want to study their relationship

- Following assumptions are made:
  - Object distance large with respect to focal length
  - E_e proportional to entrance pupil
  - E_e inversionally proportonal to square of focal length f^2. This because lateral magnification is proportional to focal length: the longer the focal length, the larger the area covered by the image

Imaging radiometry

- Differential area *dA*, off-axis in the object plane, projecting to a corresponding differential area *dA′* on image plane
- Between these areas there is the optical system
- Chief ray from dA makes angle θ with optical axis.
- s distance dA entrance pupil
- h: distance from optical axis
- d: radius entrance pupil
- dΨ: diff. area on entrance pupil at distance r from optical axis



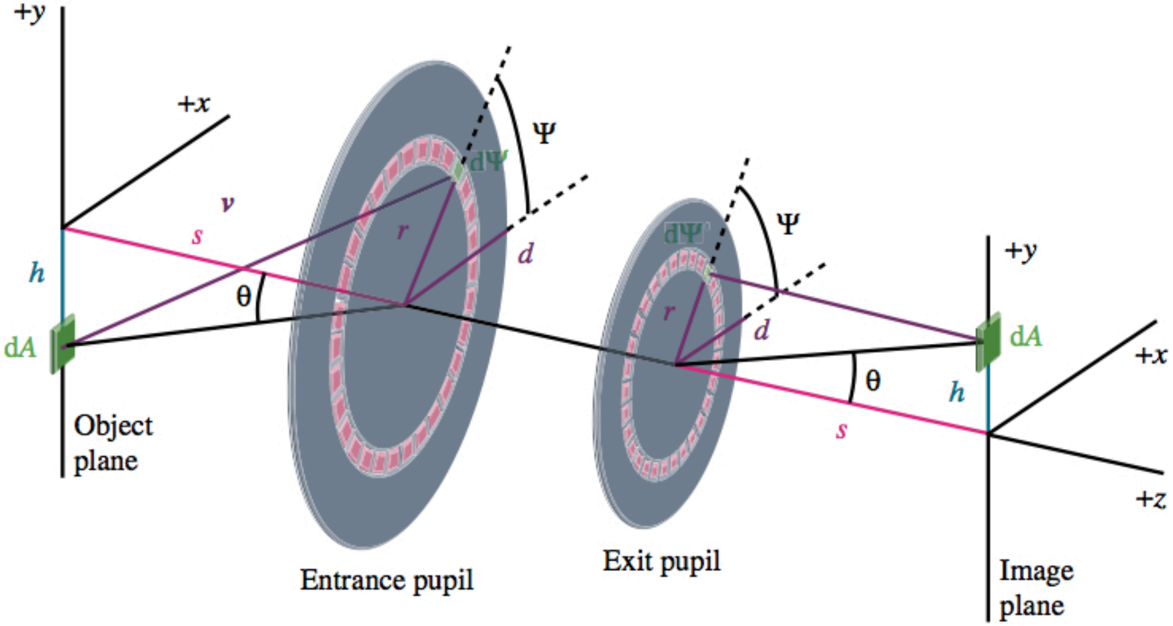
Imaging radiometry

- We want to integrate over entrance pupil, i.e. sum dΨ
- Vector v from dA to dΨ:

$$\mathbf{v} = \begin{bmatrix} r\cos(\Psi) \\ r\sin(\Psi) - h \\ s \end{bmatrix}$$

- v makes an angle α with optical axis, computable from

$$\cos(\alpha) = \frac{s}{\|\mathbf{v}\|}$$



Imaging radiometry

- If dA is lambertian then the flux incident into dA’ is

$$d\Phi_0 = L_e \int_{r=0}^{d} \int_{\Psi=0}^{2\pi} \frac{r\, d\Psi\, dr\, \frac{s}{\|\mathbf{v}\|}}{\|\mathbf{v}\|^2}\, dA\, \frac{s}{\|\mathbf{v}\|}$$

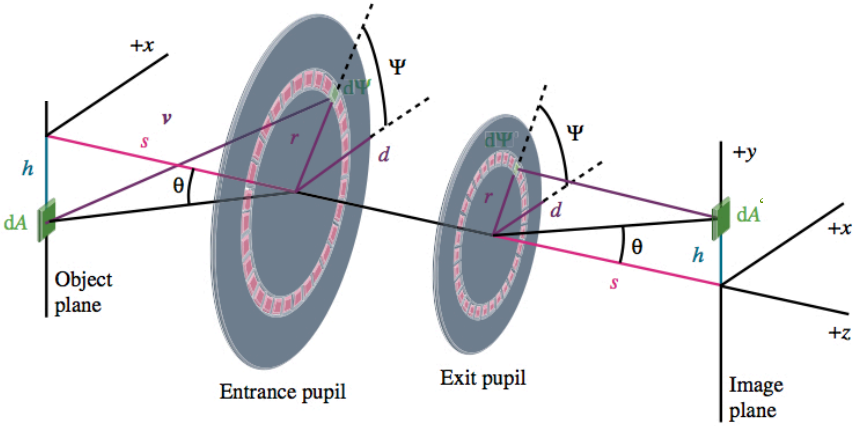
$$= L_e \int_{r=0}^{d} \int_{\Psi=0}^{2\pi} \frac{r s^2\, d\Psi\, dr}{\left(r^2 \cos^2(\Psi) + (r \sin(\Psi) - h)^2 + s^2\right)^2}\, dA$$

$$= L_e\, dA \int_{r=0}^{d} \frac{2\pi \left(s^2 + h^2 + r^2\right) r s^2 dr}{\left((s^2 + h^2 + r^2)^2 - 4h^2 r^2\right)^{3/2}}$$

$$= \frac{\pi}{2} L_e\, dA \left(1 - \frac{s^2 + h^2 - d^2}{\left((s^2 + h^2 + d^2)^2 - 4h^2 d^2\right)^{1/2}}\right).$$

- Similarly for quantities at the exit pupil (indicated with ‘)

$$d\Phi_1 = \frac{\pi}{2} L_e'\, dA' \left(1 - \frac{s'^2 + h'^2 - d'^2}{\left((s'^2 + h'^2 + d'^2)^2 - 4h'^2 d'^2\right)^{1/2}}\right)$$



Imaging radiometry

- If the optical system has no light losses, flux at entrance and exit pupils are the same:

$$E'_e = \frac{d\Phi_0}{dA'}$$

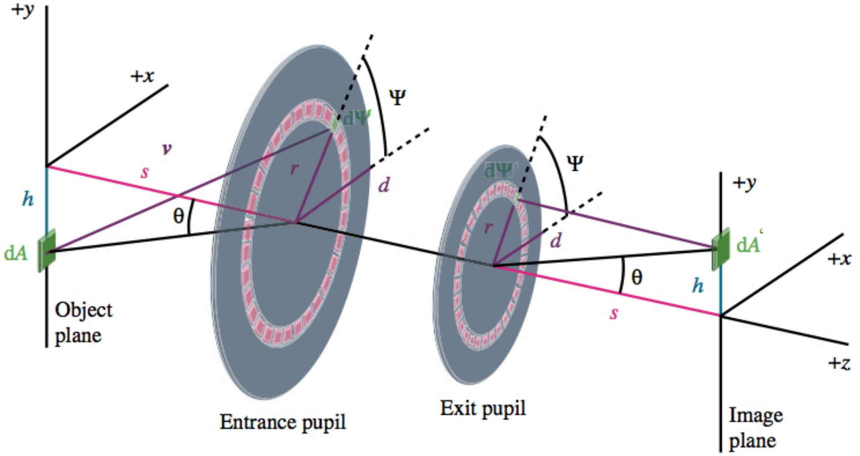
$$= \frac{\pi}{2} L_e \frac{dA}{dA'} \left(1 - \frac{s^2 + h^2 - d^2}{\left((s^2 + h^2 + d^2)^2 - 4h^2 d^2 \right)^{1/2}} \right)$$

- This is equivalent to

$$= \frac{\pi}{2} L'_e \left(1 - \frac{s'^2 + h'^2 - d'^2}{\left((s'^2 + h'^2 + d'^2)^2 - 4h'^2 d'^2 \right)^{1/2}} \right)$$

- Similarly for quantities at the exit pupil (indicated with ‘)

$$d\Phi_1 = \frac{\pi}{2} L'_e \, dA' \left(1 - \frac{s'^2 + h'^2 - d'^2}{\left((s'^2 + h'^2 + d'^2)^2 - 4h'^2 d'^2 \right)^{1/2}} \right)$$



Imaging radiometry

- Call:
    - n refraction index at object plane
    - n’ refraction index at image plane
- Then:

$$E'_e = \frac{\pi}{2} L'_e \left(\frac{n'}{n}\right)^2 \left(1 - \frac{s'^2 + h'^2 - d'^2}{\left((s'^2 + h'^2 + d'^2)^2 - 4h'^2 d'^2\right)^{1/2}}\right)$$

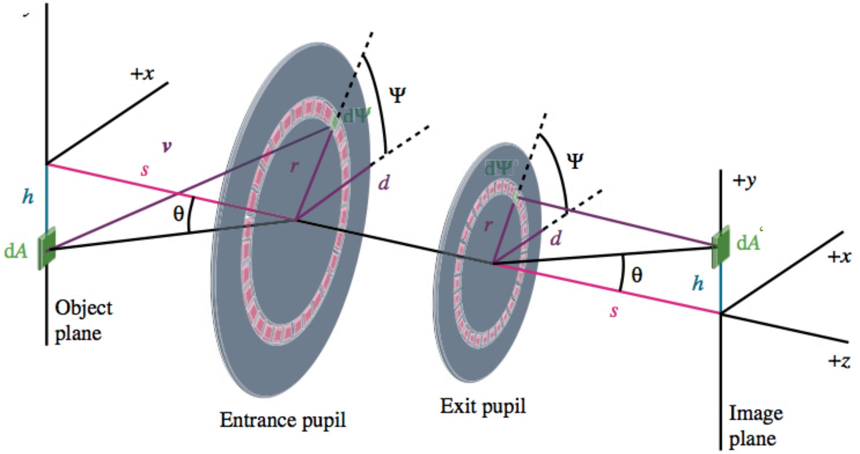
$$= \frac{\pi}{2} L'_e \left(\frac{n'}{n}\right)^2 G,$$

where

$$G = 1 - \frac{s'^2 + h'^2 - d'^2}{\left((s'^2 + h'^2 + d'^2)^2 - 4h'^2 d'^2\right)^{1/2}}$$

Image Irradiance Equation

- IIE is general, but hard to compute
- It can be simplified for certain cases: for example for on-axis imaging, as well as for off-axis imaging
    - Object distance much larger than entrance pupil



On axis image irradiance

- When object of interest is on optical axis, then h=h'=0.
  The equation simplifies to:

$$E'_e = \pi L_e \left(\frac{n'}{n}\right)^2 \left(\frac{d'^2}{s'^2 + d'^2}\right)$$

  Consider the cone spanned by the exit pupil as the base and the on-axis point on the image plane as the apex:
- then the sine of the half-angle β of this cone is given by:

$$\sin(\beta) = \frac{d'^2}{\sqrt{s'^2 + d'^2}}$$

  substituting:

$$E'_e = \frac{\pi L_e}{n^2}\left(n' \sin(\beta)\right)^2$$

- n' sin(β) is called *numerical aperture*
- E'_e is proportional to numerical aperture: the larger the aperture, the lighter the image (speed of system)
- A related measure is the *relative aperture* F (*f-number*):

$$F = \frac{1}{2n' \sin(\beta)}$$

- If image point at infinity, then one can assume distance between image plane and exit pupil *s′* = image focal length *f′*
- And $\beta \approx \tan^{-1}(d'/f')$ so relative aperture becomes

$$F_\infty \approx \frac{1}{2n' \sin\left(\tan^{-1}\left(d'/f'\right)\right)} \approx \frac{1}{n'}\frac{f'}{2d'}.$$

On axis image irradiance

- Using pupil magnification m_d=d/d' we can rewrite as

$$F_\infty \approx \frac{1}{m_p n} \frac{f}{2d}$$

if object and image plane are in air, then refraction index is 1

- If magnification factor is close to 1, then relative aperture for object at infinity can be approximated:

$$F_\infty = \frac{f}{D}$$

where D=diameter of entrance pupil

- An alternative notation for the f-number is f/N, where N is replaced by f/D
- So, for a lens of focal length of 50mm and aperture of 8.9mm, the f-number is written as f/5.6.
- Immage irradiance can be written as:

$$E'_e = \frac{\pi D^2 L_e}{4} \left(\frac{m_p}{f} \right)^2$$

notice: $\pi D^2/4$ = area of entrance pupil

Off axis image irradiance

- For objects not on optical axis we can assume distance to entrance pupil much bigger than entrance pupil radius (s≫d): irradiance is approximated as:

$$E'_e \approx \pi L_e \frac{s^2 d^2}{(s^2+d^2+h^2)^2}\frac{dA}{dA'}$$

$$\approx \pi L_e \frac{s^2 d^2}{(s^2+h^2)^2}\frac{dA}{dA'}.$$

look at picture: cosine of off axis angle θ is $\cos(\theta) = \frac{s}{\sqrt{s^2+h^2}}$

thus image irradiance becomes

$$E'_e \approx \pi L_e \cos^4(\theta)\left(\frac{d}{s}\right)^2 \frac{dA}{dA'}$$

now dA/dA’ is related to lateral magnification of the lens m through

$$m = \sqrt{\frac{dA}{dA'}}$$

- So: $E'_e \approx \pi L_e \cos^4(\theta)\left(\frac{d}{s}\right)^2 m^2.$

lateral magnification satisfies $\frac{m}{m-1} = \frac{f'}{s}$ thus

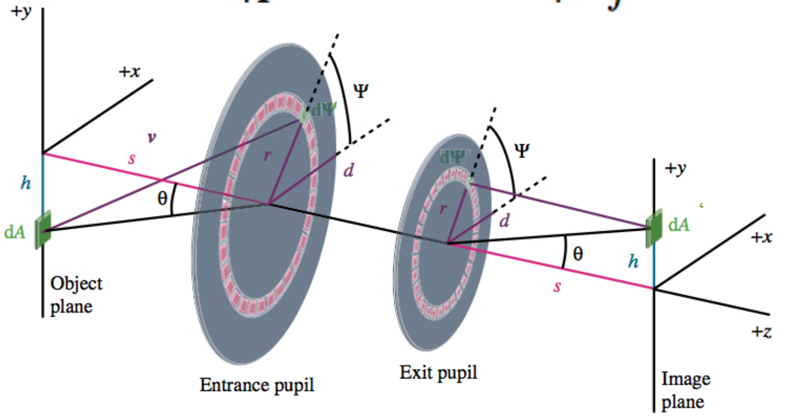
$$E'_e \approx \pi L_e \cos^4(\theta)\left(\frac{d}{(m-1)\,f'}\right)^2$$

or in terms of f-number

$$E'_e \approx \frac{\pi L_e}{4F^2\,n'^2}\,\frac{1}{(m-1)^2\,m_p^2}\cos^4(\Theta)$$

for m=2, m_p=1, refraction at image is 1, so the falloff is $\cos^4$

$$E'_e \approx \frac{\pi L_e}{4F^2}\cos^4(\Theta) \approx \frac{\pi L_e}{4}\frac{d^2}{f^2}\cos^4(\Theta)$$

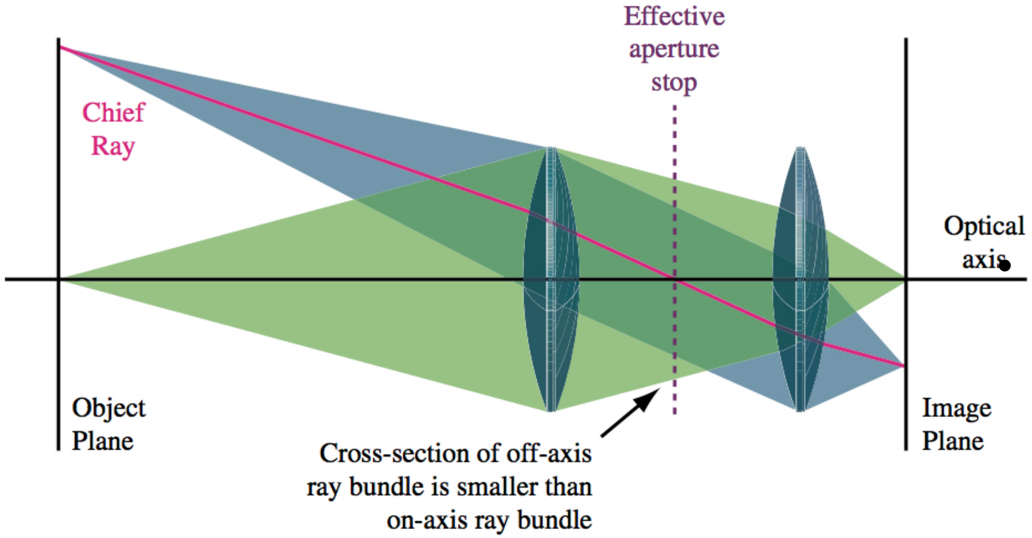


Off axis image irradiance

- The consequence? Light falloff!
- Modern lenses tend to perform better than $\cos^4$

Vignetting

- For simple opt.sys. as in picture the dimension of lenses impose an aperture
- The cross-section of aperture depends on which point in object plane is used
- Further off axis=smaller cross-section



- So, less light arrives to image space, so additional fall-off called *vignetting*
- Amount depends on distance to optical axis
- We introduce
  - spatial dependency on points in the object plane (*x*,*y*) and corresponding points on the image plane (*x*',*y*'),
  - attenuation factor *V* (*x*′ , *y*′) that takes vignetting into consideration
- Irradiance becomes:

$$E'_e(x',y') = \frac{\pi}{2} L'_e(x',y')\; T\; V(x',y') \left(\frac{n'}{n}\right)^2 G$$

Glare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Glare

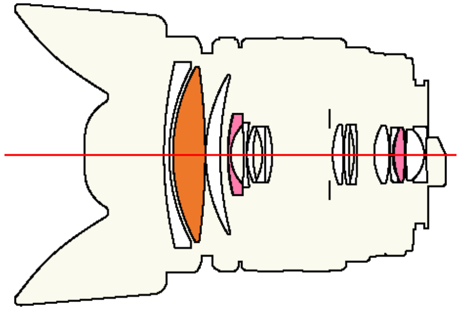
- Optical systems have many imperfections not taken into account by the irradiance equation
- Lens barrel and aperture blades might scatter light, so some light will be smeared all over the image plane: veiling glare or lens flare
- Frequent by looking at light sources
- Others might result from reflections inside the lens

- Modeling glare for the irradiance can be done by adding a glare function g(x’,y’)

$$E'_e(x',y') = \frac{\pi}{2} L'_e(x',y')\, T\, V(x',y') \left(\frac{n'}{n}\right)^2 G + g(x',y').$$

  the more components a lens has, the more prone it is to glare

- Especially true in zoom lenses



End

- Thank you for listening!