Proceedings of the 5th International Conference on
Computation of Shell and Spatial Structures
June 1-4, 2005 Salzburg, Austria
E. Ramm, W. A. Wall, K.-U. Bletzinger, M. Bischoff (eds.)
www.iassiacm2005.de

Stochastic dynamic stability analysis of shell structures

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Abstract

This document presents an analysis method to investigate the dynamic stability behavior of random geometrically imperfect systems loaded by random excitations. The analysis uses a time integration method to consider the nonlinearities of the structures and is based on the stability concept of Lyapunov. The numerical results are compared with the results of a well known linear analysis method by analyzing a nonlinear multi-degree-of-freedom shell structure.

1 Introduction

This paper gives a very brief overview of our investigations in stochastic dynamic stability analysis. More detailed informations can be found in Most et al. [8]. In this analysis geometrical imperfections are constructed as randomly spatially distributed deviations from a perfect geometry. Mathematically these imperfections are modeled as random fields which are discretized at the nodes of the finite element model. The eigenvectors of the covariance matrix of the random field obtained by a diagonalization of this matrix (Ghanem and Spanos [6]) can be interpreted as orthogonal imperfection shapes with probabilistic weights. The stability behavior of geometrically imperfect systems are analyzed separately for each shape by using standard methods of structural mechanics.

The method of analysis is based on the convergence criterion "stability with probability one". To analyze the stability a time integration of the system with an accompanying stability analysis until infinity is theoretically required, see e.g. Eller [5]. Principally all nonlinearities of the system can be considered if the nonlinear system matrices are computed time-step-wise. This time integration is the crucial numerical operation. Implicit time integration methods of the Newmark type can fail due to ill-conditioned stiffness matrices in the vicinity of the stability border. An explicit time integration method is applied here which is limited by a system-dependent critical time step.

2 Probabilistic model

Geometrical imperfections are interpreted as spatially fluctuating structural properties with respect to a perfect geometry and are described by random fields with a defined degree of homogeneity and isotropy (Vanmarcke [12]). In this paper the imperfections are assumed to be weakly homogeneous and normally distributed, characterized by an exponential correlation function with a defined correlation length l_h.

By discretising the random field using the nodes of a finite element structure, the correlation matrix can be obtained as a function of the nodal coordinates (Brenner [2]). The random field is conditioned by assuming the support conditions as deterministic. The modified *conditional* random field (Vanmarcke [12], Ditlevsen [4]) is no longer weakly homogeneous. Its parameters are determined via a stochastic interpolation scheme which is based on the maximum likelihood principle (Ditlevsen [4]).

The final correlation matrix $\hat{\mathbf{C}}_{xx}$ is diagonalized:

$$\mathbf{\Psi}^T \hat{\mathbf{C}}_{xx} \mathbf{\Psi} = diag\left(\sigma^2_{Y_i}\right) \text{ with } \sigma^2_{Y_1} \geq \sigma^2_{Y_2} \geq ...\sigma^2_{Y_n} \tag{1}$$

The eigenvectors $\mathbf{\Psi}$ characterize the imperfection shapes, the eigenvalues $\sigma^2_{Y_i}$ represent the variances of the respective amplitudes. These amplitudes are normally distributed, have zero mean and are ordered with decreasing size (Brenner [2]).

The failure probability of the structure is computed by integration of the marginal distribution of the random variable vector $\mathbf{Y}$ over the failure domain indicated by $g(\mathbf{y}) \leq 0$:

$$p_f = \int_{g(\mathbf{y})<0} f_{\mathbf{Y}}(\mathbf{y}) d\mathbf{y}, \tag{2}$$

where $g(\mathbf{y}) \leq 0$ indicates the region of instability. To solve Eq.2 imperfection shapes are increased until the stability border is reached. When $f_{\mathbf{Y}}$ is of dimension one the failure probability can be obtained analytically. To investigate a higher dimensional problem an interaction model can be applied.

3 Mechanical model

3.1 Reference solution and consistent linearization

The nonlinear equation of motion of a system can be written in a matrix-vector equation:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{r}(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{f} \tag{3}$$

with the mass matrix $\mathbf{M}$, the nonlinear restoring force vector $\mathbf{r}$ depending on the nodal displacement vector $\mathbf{x}$ and the time depending continuous loading function $\mathbf{f}$. Eq.3 is valid for any perfect or imperfect structural system. The formal linearization of the nonlinear restoring forces, which are supposed to be continuous and differentiable, with respect to a continuous reference solution $\mathbf{x}_0$ and $\dot{\mathbf{x}}_0$ yields:

$$\mathbf{r} = \mathbf{r}\left(\mathbf{x}_0, \dot{\mathbf{x}}_0\right) + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} \tag{4}$$

with the deviation from the reference solution $\mathbf{y} = \mathbf{x} - \mathbf{x_0}$ and the tangential stiffness and the damping matrices $\mathbf{K}$ and $\mathbf{C}$ respectively. The equation of motion can be split into a differential equation for the reference solution itself,

$$\mathbf{M}\ddot{\mathbf{x}}_0 + \mathbf{r}(\mathbf{x}_0, \dot{\mathbf{x}}_0) = \mathbf{f} \tag{5}$$

and a differential equation for the difference to neighboring motions:

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{0} \tag{6}$$

3.2 Nonlinear stability analysis

To analyze the dynamic stability behavior of nonlinear systems an integration of Eq.5 is necessary until stochastic stationarity is reached. In each time step, the tangential stiffness matrix $\mathbf{K}$ has to be determined. With this kind of analysis a criterion for sample stability is developed. In order to speed up explicit time integration, this equation can be projected into a subspace of dimension m as spanned by the eigenvectors of the undamped system corresponding to the m smallest natural frequencies (Bucher [3]). These eigenvectors are the solutions to

$$\left(\mathbf{K}\left(\mathbf{x}_{stat}\right) - \omega_i^2 \mathbf{M}\right) \mathbf{\Phi} = 0; \quad i = 1 \dots m \tag{7}$$

In this equation $\mathbf{x}_{stat}$ is chosen to be the displacement solution of Eq.5 under static loading conditions. The mode shapes are assumed to be mass normalized. A transformation $\mathbf{x} = \mathbf{\Phi v}$ and a multiplication of Eq.5 with $\mathbf{\Phi}^T$ represents a projection of the differential equation of motion for the reference solution into the subspace of dimension m as spanned by the eigenvectors:

$$\ddot{\mathbf{v}} + \mathbf{\Phi}^T \mathbf{r}(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{\Phi}^T \mathbf{f} \tag{8}$$

The integration of Eq.8 by the central difference method (Bathe [1]) requires a minimal time step.

The time integration in the subspace and the computing of the restoring forces on the full system causes the following problem: If the initial displacement or velocity vector of the time integration is not zero, for example due to static loading, the projection of these vectors into the subspace is an optimization problem caused by the higher number of variables in the full space. A possibility to improve the situation is given in Most et al. [8].

To analyze the stability behavior of the reference solution $\mathbf{x}_0(t)$, the long-term behavior of the neighboring motion (Eq.6) is investigated. To reduce the dimension of the equation system, this equation can be projected into the same or a smaller subspace as Eq.8. Transformed into the state space description we obtain:

$$\dot{\mathbf{z}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{\Phi}^T \mathbf{K \Phi} & -\mathbf{\Phi}^T \mathbf{C \Phi} \end{bmatrix} \mathbf{z} = \mathbf{A}[\mathbf{x}_0(t)]\mathbf{z} \tag{9}$$

From this equation the Lyapunov exponent λ can be determined by a limiting process:

$$\lambda(\mathbf{x}_0, \mathbf{s}) = \lim_{t \to \infty} \frac{1}{t} \log ||\mathbf{\Theta}(\mathbf{x}_0, t)\mathbf{s}|| \tag{10}$$

in which $\mathbf{s}$ is an arbitrary unit vector. In Eq.10 $\mathbf{\Theta}(\mathbf{x}_0, t)$ is the transition matrix from time 0 to t associated with Eq.9. In the current investigation, the norm $||\mathbf{\Theta}(\mathbf{x}_0, t)\mathbf{s}||$ is expressed in terms of

$$\|\mathbf{\Theta}(\mathbf{x}_0, t)\mathbf{s}\| \leq \|\mathbf{\Theta}(\mathbf{x}_0, t)\| \cdot \|\mathbf{s}\| = \|\mathbf{\Theta}(\mathbf{x}_0, t)\| \tag{11}$$

Finally, this result is used in calculating the Lyapunov exponent according to Eq.10 by using a matrix norm equal to the eigenvalue μ_{max} of $\mathbf{\Theta}(\mathbf{x}_0, t)$ with the maximum absolute value. The time domain t has to be taken large enough that the Lyapunov exponent convergences to a stationary value.

4 Numerical example: Reliability investigation of a shell structure

A cylindrical panel was considered, which is mentioned e.g. in Krätzig [7] and Schorling and Bucher [10]. The assumed structure is shown in Fig.1. The geometrical and the material properties were given as: radius $R = 83.33m$, the half width and height $a = 5m$, the thickness $h = 0.1m$, the Young's modulus $E = 3.410^{10} N/m^2$, the mass density $\rho = 3400 kg/m^3$ and the Poisson's ratio $\mu = 0.2$. The constant load factor is $P = 1000N/m$.

The structure is discretized with 7×7 nodes and meshed with geometrically nonlinear 9-node shell elements. At a static load factor of $\nu_{crit} = 16825$ the structure reaches an unstable state (Krätzig [7]: $\nu_{crit} = 15120$, Schorling and Bucher [10]: $\nu_{crit} = 16200$; both used a different discretization and different elements). The static load is assumed to be $P_0 = 0.85\nu_{crit}P$. The fluctuating load is considered as $P_{fluct} = \ell f(t)P$, where $f(t)$ is a unit white noise process and ℓ is the load factor. The damping is assumed as modal damping with the damping ratio $\zeta_k = 0.02$ for all modes. The random excitation process is discretized by using Fourier series (FFT) with Fourier coefficients as zero-mean Gaussian random variables and random amplitudes according to Rice [9].

The geometrical imperfections are considered in terms of radial deviations from the perfect panel surface and are modeled as a conditional Gaussian random field. The mean is assumed as zero and the standard deviation as $\sigma = 10^{-3}m$. The correlation length of the exponential correlation function is considered with $l_H = 10m$. The imperfection shapes are obtained by the decomposition of the covariance matrix according to Eq.1. The first four imperfection shapes are shown in Fig.2. The corresponding standard deviations σ_{Yi} in uncorrelated normal space are indicated in the figure. The first shape is very similar to the buckling shape.

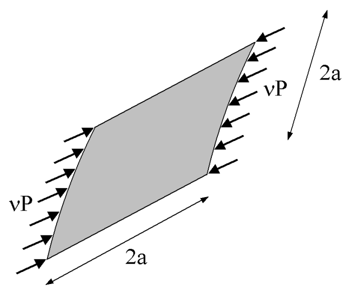

Figure 1: Cylindrical shell structure

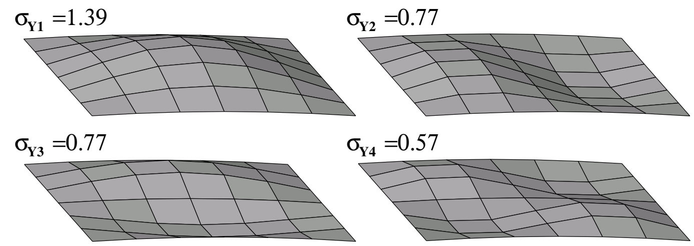

Figure 2: Weighted imperfection shapes

The structure was investigated by using the linear Itô analysis (Soong and Grigoriu [11]) and it was found that only the first imperfection shape has a major influence on the stability behavior. The critical noise intensity of the perfect system was obtained as $D_{0_{crit}} = 43946\pi$ with the linear and $D_{0_{crit}} = 20357\pi$ with the nonlinear method by averaging 20 simulations with 10^5 time steps. The nonlinear analysis uses a modal subspace spanned by 12 of the 213 eigenmodes with a critical time step of $\Delta t = 6.3 \cdot 10^{-3}s$. The investigation of the first imperfection shape obtained by nonlinear analysis show observable deviations from the linear results. This points out that the nonlinearities of this structure have a high influence on the stability behavior. The obtained stability boundaries depending on the imperfection size are displayed in Fig.3 for the linear and the nonlinear analysis.

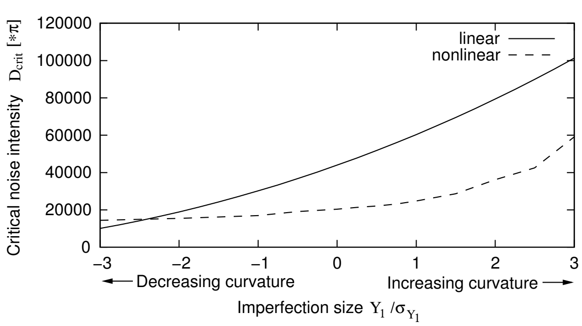

Figure 3: Stability boundaries vs. imperfection size

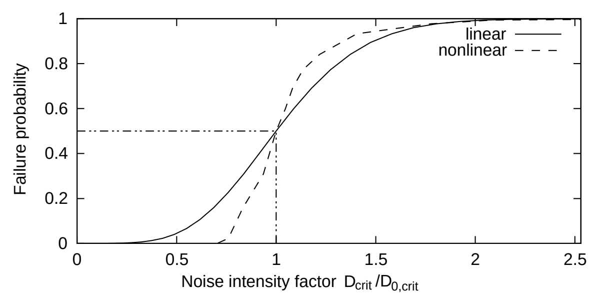

Figure 4: Failure probabilities using linear and nonlinear analyses

The failure probability for this one dimensional problem can be obtained analytical from the stability boundaries and is shown in Fig.4 depending on the noise intensity for both methods. It is to be seen in the picture, that a sufficient approximation of the nonlinear probability graph is not possible with the linear method.

5 Conclusions

The paper presents a nonlinear method to analyze the stochastic dynamic stability of shell structures. This method can consider geometrical and material nonlinearities by using an explicit time integration. For reliability analysis the random imperfections are represented by a conditional random field, whose correlation matrix is diagonalized in different imperfection shapes and respective random amplitudes. The failure probability is computed depending on the dimension of the random variable vector.

It is necessary to compute many stability boundaries from different imperfection-size/excitation combinations for each imperfection shape. This high number of simulations is not realizable for larger systems caused by the huge numerical effort. It is suggested to use a linear method to find the imperfections shapes which influence the stability behavior most significantly. The stability boundaries can be approximated with the linear method, but the nonlinear method should by used to validate the results.

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