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Stability analysis for imperfect systems with random loading

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ABSTRACT: This paper shows an extension of research activities to stochastic dynamic stability problems. Both, the structure and the loading conditions are supposed to be random. The structures investigated are subjected to fluctuating loads in the vicinity of the static buckling load. The fluctuating components of the loading are modeled by stationary random processes, described by their power spectral density. As the loading is time dependent, the top Lyapunov exponent of the system has to be derived to determine the stability of the system. The top Lyapunov exponent can either be determined by nonlinear time integration of the system with accompanying stability analysis or by linear Itô analysis. The determination of the Lyapunov exponent by time integration shows one major difficulty: In statistical sense, the statement with respect to the Lyapunov exponent gets less precise as the structural response gets more critical. The Itô calculus principally represents an efficient linear analysis method to determine the second moment stability behavior of the structural system, suitable for Finite Element analysis. Within this paper both approaches are performed and compared for geometrically perfect and imperfect systems.

1 INTRODUCTION

This paper shows an extension of recent investigations in stochastic dynamic stability analysis. The authors considered geometrically imperfect structures with static loading (Schorling 1997), periodic loading conditions (Schorling and Bucher 1998, Schorling and Bucher 1999) and for random loading (Schorling, Bucher, and Purkert 1998). Geometrical uncertainties may be interpreted as randomly spatially distributed deviations from a perfect geometrical shape. Mathematically they are covered by point discretized random fields. The covariance matrix obtained can be diagonalized (Ghanem and Spanos 1991). The eigenvectors resulting from this transformation may be interpreted as orthogonal imperfection shapes which have probabilistic weights. Their influence on the dynamic stability behavior can be analyzed by standard methods of structural mechanics.

The random loading is described by a scalar-valued random process. It is assumed to be stationary in time and normally distributed with a given mean value and power spectral density. It is represented in terms of a finite Fourier series with random coefficients (Rice 1948).

Within this paper two analysis methods to determine the stability of the structure are presented and discussed. These methods base on different convergence criterions for asymptotic stability.

The first method presented bases on the convergence criterion "stability with probability one (almost sure stability)". The stability of the structure is determined by analyzing the tangential equations of motion of the structure, see e.g. Burmeister 1987, Eller 1988, Krätzig and Nawrotzki 1996. This procedure theoretically requires a time integration of the system with an accompanying stability analysis until infinity. For this analysis type principally all nonlinearities of the system can be considered. Obviously for this method the time integration of the system is the crucial numerical operation. Further, as the structures investigated are subjected to loads which are in the vicinity of the static buckling load, implicit time integration methods of the Newmark type can fail due to ill-conditioned stiffness matrices. Here an improved and stabilized explicit time integration method is applied.

The second method presented is based on a convergence criterion of the stability expressed in term of second moments (mean square stability). The Lyapunov exponents are derived by Itô calculus, see e.g. Soong and Grigoriu 1992. This method requires an expansion of the stiffness matrix of the system into a series. When only first order terms are considered a linear equation system with parametric load effects can be established for which the Itô calculus can be performed straight forward. This method is easily applicable for finite element analysis and does not require an extensive time integration procedure. In general the stability criterion "stability in second moments" is stricter then the criterion with "probability one".

Within the examples of this paper both methods are discussed. First a single degree of freedom system is considered to analyze the principal differences and possible pitfalls of both methods, then a multi-degree of freedom finite element system is considered. Within this example the dependence of the stability behavior on the intensity of the dynamic loading and the geometrical imperfections is investigated in detail. The probabilistic and structural analysis tasks are performed with the SL$^{\mathrm{ang}}$ Software package (Bucher, Schorling, and Wall 1995, Bucher and Schorling 1997).

2 METHOD OF ANALYSIS

2.1 *Probabilistic model*

2.1.1 *Random imperfections*

Geometrical imperfections are interpreted as spatially fluctuating structural properties with respect to a perfect geometry. They are modeled as random fields, described by a mean and covariance function and a defined degree of homogeneity and isotropy (Vanmarcke 1983). For simplicity the random imperfections considered in this paper are assumed to be weakly homogeneous and normally distributed. An exponential correlation function with a defined correlation length l_h is used.

If the random field is discretized at the nodes of a finite element structure the correlation matrix may be determined in a straightforward manner as a function of the nodal coordinates (Brenner 1995). Support conditions of the structure have considerable influence on the stability behavior. In order to isolate such effects the location of the supports is assumed to be deterministic, while the structure itself remains geometrically imperfect. Mathematically this step is carried out by conditioning the random field. The resulting *conditional* random field (Vanmarcke 1983, Ditlevsen 1991) then has vanishing variances at the supports and non vanishing variances in the remaining structure and thus is non longer weakly homogeneous. Its parameters are determined via a stochastic interpolation scheme which is based on the maximum likelihood principle (Ditlevsen 1991).

2.1.2 *Random excitation*

It is assumed that the structure under investigation is excited by a stationary random process $f(t)$ with given mean value $\bar{f}$ and power spectral density $S_{ff}(\omega)$. In discrete form, $f(t)$ is represented by a Fourier series (using FFT)

$$f_n(t) = \sum_{k=1}^{n} \sigma_k (A_k \cos \omega_k t + B_k \sin \omega_k t) \tag{1}$$

The Fourier coefficients A_k, B_k are zero-mean Gaussian random variables with unit standard deviation and $\sigma_k^2 = \int_{\Delta\omega_k} S_{ff}(\omega) d\omega \approx S_{ff}(\omega_k)\Delta\omega_k$ (Rice 1948). By digital generation of realizations of A_k, B_k and subsequent application of Eq. 1 sample functions of $f(t)$ are obtained.

2.2 *Mechanical model*

2.2.1 *Reference solution and consistent linearization*

Structural dynamic response analysis requires the solution of the following matrix-vector equation:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{r}(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{f} \tag{2}$$

M represents the mass matrix, **x** denotes the vector of nodal displacements, **f** a time depending continuous loading function and **r** the nonlinear restoring forces, which are supposed to be continuous and differentiable with respect to **x** and $\dot{\mathbf{x}}$. Vectors and matrices have sizes N and $N \times N$, respectively. Eq. 2 is valid for any structural system, no matter whether it is perfect or not. The formal linearization of the nonlinear restoring forces (Eq. 2) with respect to a continuous reference solution $\mathbf{x}_0$ and $\dot{\mathbf{x}}_0$ yields:

$$\mathbf{r} = \mathbf{r}(\mathbf{x}_0, \dot{\mathbf{x}}_0) + \left.\frac{\partial \mathbf{r}}{\partial \dot{\mathbf{x}}}\right|_{\mathbf{x}_0, \dot{\mathbf{x}}_0} \dot{\mathbf{y}} + \left.\frac{\partial \mathbf{r}}{\partial \mathbf{x}}\right|_{\mathbf{x}_0, \dot{\mathbf{x}}_0} \mathbf{y}, \tag{3}$$

or

$$\mathbf{r} = \mathbf{r}(\mathbf{x}_0, \dot{\mathbf{x}}_0) + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} \tag{4}$$

where $\mathbf{y} = \mathbf{x} - \mathbf{x_0}$ denotes the deviation from the reference solution, **K** and **C** are the tangential stiffness and the damping matrices of the system considered. This enables to split the equation of motion into a differential equation for the reference solution itself,

$$\mathbf{M}\ddot{\mathbf{x}}_0 + \mathbf{r}(\mathbf{x}_0, \dot{\mathbf{x}}_0) = \mathbf{f} \tag{5}$$

and a differential equation for the difference to neighboring motions:

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{0} \tag{6}$$

2.2.2 *Nonlinear stability analysis*

The treatment of the dynamic stability problem for stochastic loading for nonlinear systems requires an integration of Eq. 5 until stochastic stationarity is reached. In each time step, the tangential stiffness matrix **K** has to be determined. With this kind of analysis a criterion for sample stability is developed. In order to speed up explicit time integration, this equation is then projected into a subspace of dimension m as spanned by the eigenvectors of the undamped system corresponding to the m smallest natural frequencies (Bucher 2001).

The stability of the reference solution $\mathbf{x}_0(t)$ is determined by the long-term behavior of the neighboring motion (Eq.6). This equation can be rewritten in first order form:

$$\dot{\mathbf{z}} = \mathbf{A}\big[\mathbf{x}_0(t)\big]\mathbf{z} \tag{7}$$

From this equation, the Lyapunov exponent λ can be determined by a limiting process:

$$\lambda(\mathbf{x}_0, \mathbf{s}) \lim_{t \to \infty} \frac{1}{t} \log \left\| \Theta(\mathbf{x}_0, t)\mathbf{s} \right\| \tag{8}$$

in which **s** is an arbitrary unit vector. In **Eq.** 8, $\Theta(\mathbf{x}_0, t)$ is the transition matrix from time 0 to t associated with Eq. (14). This matrix is approximated by the following product of random matrices:

$$\Theta(t, 0) = \prod_{n=1,n\ step} \Theta(t_n, t_{n-1}) \tag{9}$$

In case the time interval $\Delta t = t_n - t_{n-1}$ is chosen small enough to consider the term $\mathbf{A}(\mathbf{t})$ constant within any interval the transition matrix between the time steps may be derived analytically (e.g. Riemer et al., 1993):

$$\Theta(t_n, t_{n-1}) = \mathbf{R}(t_{n-1}) diag(e^{\kappa_i \Delta t}) \mathbf{R}(t_{n-1})^{-1} \tag{10}$$

Based on the multiplicative ergodic theorem (e.g. Arnold and Imkeller 1994) the Lyapunov exponent can also be calculated as an expected value:

$$\lambda(\mathbf{x}_0, \mathbf{s}) = E\Big[\frac{d}{dt} \log \big\|\Theta(\mathbf{x}_0, t)\mathbf{s}\big\|\Big] \tag{11}$$

In the current investigation, the norm $\|\Theta(\mathbf{x}_0, t)\mathbf{s}\|$ is expressed in terms of

$$\big\|\Theta(\mathbf{x}_0, t)\mathbf{s}\big\| \leq \big\|\Theta(\mathbf{x}_0, t)\big\| \cdot \|\mathbf{s}\| = \big\|\Theta(\mathbf{x}_0, t)\big\| \tag{12}$$

In this equation, a matrix norm must be chosen which is compatible to the vector norm used in Eq.(8). If the Euclidean vector norm is used, equality in Eq. (12) is obtained by choosing the matrix norm equal to the eigenvalue μ_{max} of $\Theta(\mathbf{x}_0, t)$ with maximum absolute value. Finally, this result is used in calculating the Lyapunov exponent according to Eq. (8). The limit for t has to be taken at some finite value for which convergence can be assumed. For the statistical estimation of the convergence of the Lyapunov exponent, Eq.(11) is suitable.

2.2.3 *Linear stability analysis*

For a linearized system the Lyapunov exponent can be determined by Itô analysis. In difference to the above mentioned procedure this type of analysis determines the stability of the second moments of the reference solution. The stiffness matrix within the equation for the neighboring motion (Eq. 6) can be expanded into an asymptotic series with respect to a static loading condition. As long as the fluctuating part is small this series can be truncated after the linear term. The equation for the neighboring motion then reads:

$$\mathbf{M\ddot{y}} + \mathbf{C\dot{y}} + \big(\mathbf{K}(\mathbf{x_{stat}}) + \mathbf{f(t)K_1}\big)\mathbf{y} = \mathbf{0} \tag{13}$$

Again, this equation is projected into a subspace of dimension m and then transformed into its state space description:

$$\dot{\mathbf{z}} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ diag(\omega^2) & \mathbf{\Phi}^T\mathbf{C}\mathbf{\Phi} \end{bmatrix}\mathbf{z} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{\Phi}^T\mathbf{K}_1\mathbf{\Phi} & \mathbf{0} \end{bmatrix}\mathbf{zf(t)} \tag{14}$$

Or abbreviated,

$$\dot{\mathbf{z}} = \big[\mathbf{A} + \mathbf{Bf(t)}\big]\mathbf{z} \tag{15}$$

In this equation the coefficient matrices **A** and **B** are constant. If the fluctuating loading function is approximated by Gaussian white noise the equation of motion for the neighboring states represents a first order stochastic differential equation. In this case the top Lyapunov exponent λ_2 for the second moments of the system is easily derived by Itô calculus (e.g. Soong and Grigoriu 1992, Lin and Cai 1995).

3 NUMERICAL EXAMPLE

3.1 *SDOF-system*

As a first example for verification of the simulation procedure consider the response of a SDOF-system under parametric excitation. The equation of motion can be written in the form of

$$m\ddot{x} + c\dot{x} + k\big[1 + \ell f(t)\big]x = 0; \quad \mathbf{A}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -\frac{k}{m} & 0 \end{bmatrix} f(t) \tag{16}$$

The random process $f(t)$ is broad band with zero mean and unit standard deviation. The load factor ℓ is increased from 0.5 to 2.0 in four steps. Numerical values of $m = 1kg$, $k = 25N/m$, $c = 0.2kg/s$ (corresponding to a modal damping ratio of $\zeta = 0.02$) are used in the analysis. Figs. 1 show the

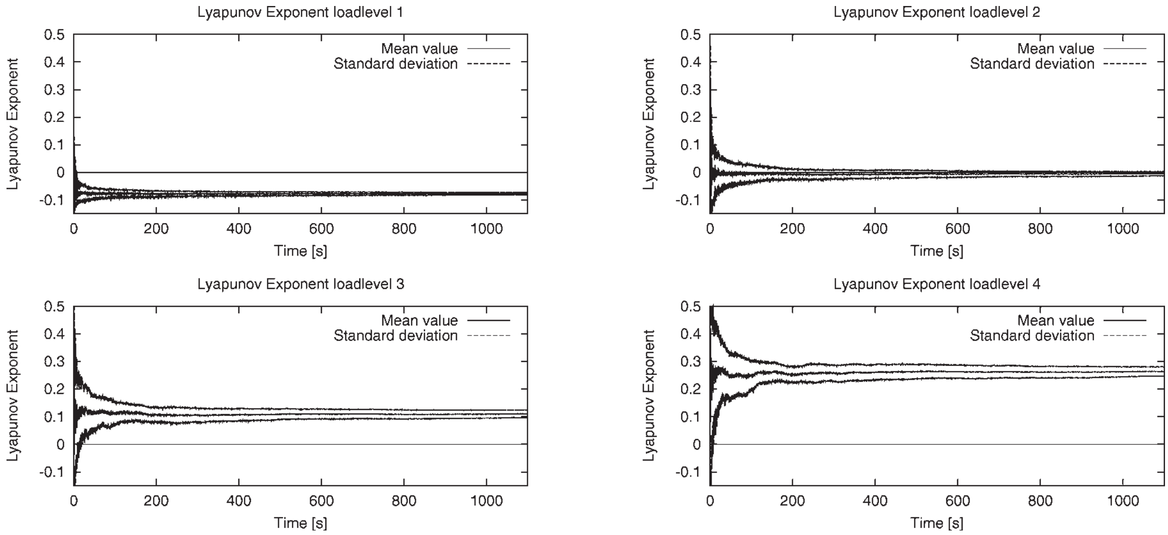

Figure 1. Lyapunov exponents for SDOF-system.

Table 1. Verification of Lyapunov exponents for SDOF-system.

Load Factor ℓ	Analytical	Numerical	Statistical Error	From λ_2
0.5	−0.0755	−0.0769	0.0045	−0.0755
1.0	−0.0018	−0.0050	0.0076	−0.0019
1.5	0.1209	0.1105	0.0139	0.1193
2.0	0.2927	0.2637	0.0185	0.2828

time-dependent mean values and standard deviations of the estimated Lyapunov exponents from 50 simulations, each having 35.000 time steps at $\Delta t = \frac{\pi}{100}s$.

The figures clearly indicate that the estimate for the Lyapunov exponent is a random quantity. This is a consequence of the band limitation (ω_{max}) and the spectral discretization ($\Delta\omega$) of the white noise. It can be observed that both the mean value $\bar{\lambda}$ and the standard deviation σ_λ are quite stable after a certain number of time steps. The standard deviation of the mean value estimate σ_{mean} from n_{sim} simulations can be calculated from $\sigma_{mean} = \frac{\sigma_\lambda}{\sqrt{n_{sim}}}$.

The example as discussed has an analytical approximate solution for the Lyapunov exponent as given by Lin and Cai 1995: $\lambda = -\zeta\omega_0 + \frac{\pi\ell^2 S_{ff}\omega_0^2}{4}$ in which $\omega_0 = \sqrt{\frac{k}{m}}$, ζ is modal damping ratio and S_{ff} is the power spectral density of the white noise. Table 1 shows the analytical vs. the numerical results. The agreement is excellent which verifies the numerical procedure. In addition, the Lyapunov exponent λ_2 for the second moments can be calulated according to $\lambda_2 = -2\zeta\omega_0 + \pi\ell^2 S_{ff}\omega_0^2$. This relation can be exploited to approximate the Lyapunov exponent for the samples from the second moment exponent according to

$$\lambda = \frac{\lambda_2}{4} - \frac{\zeta\omega_0}{2} \qquad (17)$$

It should be mentioned that the term $-\zeta\omega_0$ in this equation corresponds to the Lyapunov exponent of the system without random parametric excitation. Results according to this relation are given in Table 1. Excellent agreement is found again.

3.2 *Simple column*

As a further example consider a simply supported column subjected to a random vertical load as shown in Fig.2. The static load F_0 is chosen to be 80% of the critical load of the perfect column. A nonlinear static stability analysis leads to the value of $F_{crit} = 259.2N$. The dynamic load is chosen

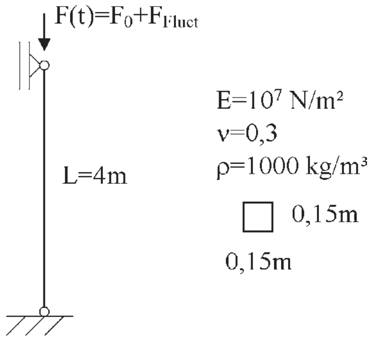

Figure 2. Column subjected to parametric excitation.

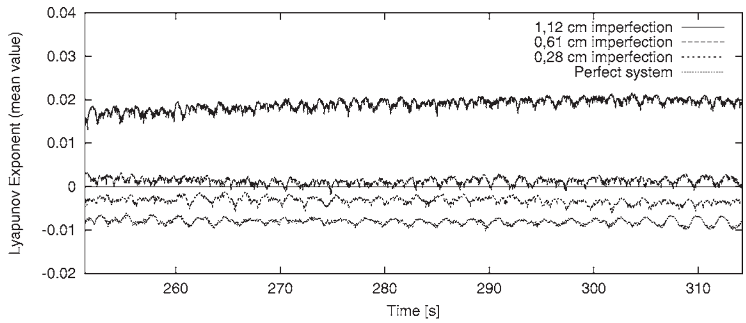

Figure 3. Mean value of Lyapunov exponent.

to be a Gaussian white noise in the form of $F_{fluct} = 300w(t)$ in which $w(t)$ is a sequence of i.i.d. random variables with zero mean and unit standard deviation. For the dynamic stability analysis, a modal damping ratio of $D_k = 0.01$ is assumed for all modes. This is used to construct the damping matrix for the nonlinear dynamic analysis. Imperfections porportional to the first buckling shape are assumed. The magnitudes of mid-span imperfections are varied from 0 to 2.79 *cm* are considered. The mean estimates for the Lyapunov exponents during a small time window from the dynamic analysis is shown in Fig.3 for some imperfection magnitudes.

The destabilizing influence of the geometrical imperfections is easily seen from the results as given in Fig.4.

The influence of the number of modes m retained throughout the modal reduction has been investigated. Fig.5 compares the Lyapunov exponents obtained from analyses with $m = 20$ and $m = 60$, respectively. The latter case corresponds to a full explicit analysis. Critical time steps were one order of magnitude apart so that $m = 20$ led to a speed-up of 10.

Finally, the Lyapunov exponent is calculated approximately from the second moment exponent as obtained from the linearized system, using Eq. (17). These results are shown in Fig.6. For the stable region, the agreement is found to be good. In fact, the stability boundary can be predicted very well.

One possible explanation for the differences at high levels of imperfections can be seen in Fig.7.

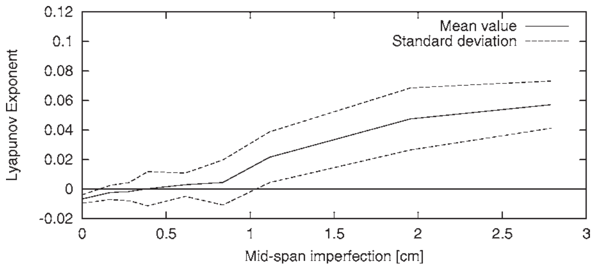

Figure 4. Mean value of Lyapunov exponent.

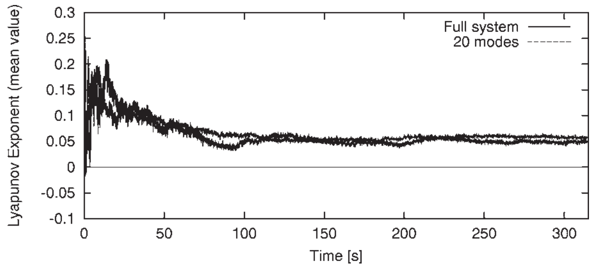

Figure 5. Influence of modal reduction on Lyapunov exponent.

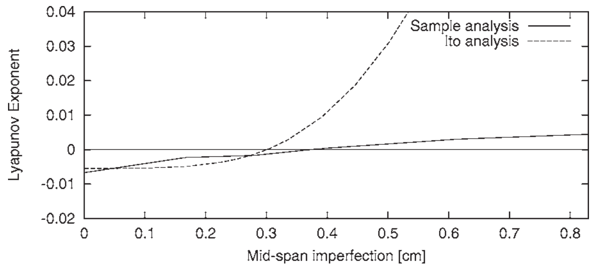

Figure 6. Comparison of Lyapunov exponents from nonlinear and linearized Itô-analysis.

Here one element of the tangential stiffness matrix is monitored throughout nonlinear dynamic analysis and compared to its linearized counterpart as obtained from Eq.(13). There is quite substantial difference in the time fluctuations.

4 SUMMARY

The paper presents an approach to consider geometric uncertainties and stochastic loading conditions within structural stability analysis tasks. The geometric uncertainties are modeled by conditional random fields, the stochastic loading conditions are described in spectral form. The random field can be divided into independent imperfection shapes with random amplitudes. These shapes can be analyzed independently from each other. The loading conditions are assumed to be ergodic. The analysis of the nonlinear dynamic stability behavior of the structure have to be performed in the time domain. Sample time series of the load process are generated by simulation procedures.

Two analysis methods are presented. For both methods the geometrical imperfections are included into the computational scheme in a similar manner.

First the nonlinear structural response due to the load processes is computed by explicit time integration schemes. The stability behavior is then computed by using the linearized equation for the neighboring motion around the reference solution. This requires an evaluation of the stiffness matrix at every time step.

The analysis task as presented shows several major difficulties. Within a reliability analysis obvi-

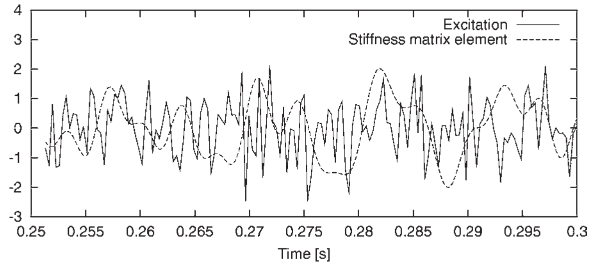

Figure 7. Comparison of linearized stiffness and nonlinear stiffness.

ously states which are in the vicinity of the limit state, this means Lyapunov exponent equal zero, are the most interesting ones. For these cases the integration by an Newmark algorithm fails, because the stiffness matrix is not necessarily longer positive definite. To overcome this difficulty we presented a suitable explicit time integration scheme. Principally the time integration has to be performed until infinity, which of course is not possible. Here a criterion has to be established which is based on the limitation of the variances for the estimator of the Lyapunov exponent, see Eq. 11. Finally and most important, in statistical sense the statement with respect to the Lyapunov exponent, gets less precise as the structural response gets more critical. Thus states which are closer to the limit state require a "longer" loading process and thus a more time consuming integration procedure.

The second analysis task represents a method based on an analytical approximation for the sample stability. This first requires a linearization of the stiffness matrix. Of course this method can be only suitable as long as such a linearization is accurate for the description of the problem. A projection into a suitable subspace is necessary, if finite element structures with a large number of degrees of freedom are considered. The method as presented in this paper is principally suitable for such type of systems. Results for the Itô analysis can be obtained in only a fraction of time, compared to the time integration methods.

Certainly the number of application for the presented analysis methods is limited "in the real world", as only few structures are submitted to a stationary loading process. Further within a design process a stability analysis is of course only one of several required analysis tasks. In especially for the case considered (stationary loading) in this paper, fatigue might be at least as relevant for the design process.

In this context it seems reasonable that for a stability investigation, first an Itô analysis is performed to check whether a design relevancy for dynamic stability and geometrical imperfections is given at all. It is emphasized again that a correct linearization of the stiffness matrix is crucial at this step. If a sensitivity is given, in a second stage the more expensive time integration method should be performed.

5 ACKNOWLEDGEMENT

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