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Stochastic dynamic stability analysis of nonlinear structures

Thomas Most & Christian Bucher
Institute of Structural Mechanics, Bauhaus-Universität Weimar, Marienstrasse 15, D-99423 Weimar, Germany

Summary This document presents an analysis method to investigate the dynamic stability behaviour of random geometrically imperfect systems loaded by random excitations. The analysis uses a time integration method to consider all nonlinearities of the structures. The method is based on the stability concept of Lyapunov. The obtained nonlinear numerical results are compared with the results of a well known linear analysis method, which is explained as well. The nonlinear and the linear method are applied on nonlinear multi-degree-of-freedom systems to compute the failure probability depending on the excitation intensity.

1 INTRODUCTION

This paper gives an short overview of the investigations in stochastic dynamic stability analysis. In previous publications, geometrically imperfect structures with static loading (Schorling 1997), periodic loading conditions (Schorling and Bucher 1998, Schorling and Bucher 1999) and for random loading (Schorling, Bucher, and Purkert 1998, Schorling, Most, and Bucher 2001) were considered. These publications construe geometrical imperfections as randomly spatially distributed deviations from a perfect geometry. Mathematically these imperfections can be modeled as random fields which are discretized by points equivalent to the nodes of the finite element model. The random field is characterized by the covariance matrix. The eigenvectors obtained by a diagonalisation of this matrix (Ghanem and Spanos 1991) can be interpreted as orthogonal imperfection shapes with probabilistic weights. The stability behaviour of geometrically imperfect systems can be analyzed separately for each shape by using standard methods of structural mechanics.

The random loading is assumed to be a scalar-valued white noise process, which can be discretized by using finite Fourier series with random coefficients (Rice 1948).

This paper presents two stability analysis methods. These methods are base on different convergence criteria for asymptotic stability. By application to different systems both methods are compared and discussed.

The first method is based on the convergence criterion "stability with probability one". To analyze the stability a time integration of the system with an accompanying stability analysis until infinity is theoretically required, see e.g. Burmeister 1987, Eller 1988, Krätzig and Nawrotzki 1996. Principally all nonlinearities of the system can be considered if the nonlinear system matrices are computed time-step-wise. This time integration is the crucial numerical operation. Implicit time integration methods of the Newmark type can fail due to ill-conditioned stiffness matrices in the vicinity of the stability border. An explicit time integration method is applied here which is limited by a system-dependent critical time step.

The second method is based on a convergence criterion of the stability expressed in term of second moments (mean square stability). The Lyapunov exponents are derived by the Itô calculus, see e.g. Soong and Grigoriu 1992. This method can be practically on linear systems. By considering only the first order terms of the asymptotic stiffness matrix series nonlinear systems can be linearized. The Itô calculus does not require an extensive time integration procedure.

Both methods are verified and applied to SDOF and MDOF systems by using the SLang Software package (Bucher et al. 1995, Bucher and Schorling 1997).

2 METHOD OF ANALYSIS

2.1 Probabilistic Model

2.1.1 Random Imperfections

To represent geometrical imperfections, which are interpreted as spatially fluctuating structural properties with respect to a perfect geometry, random fields with a defined degree of homogenity and isotropy (Vanmarcke 1983) can be applied. In this paper the imperfections are assumed to be weakly homogeneous and normally distributed, characterized by an exponential correlation function with a defined correlation length l_h.

By discretising the random field using the nodes of a finite element structure, the correlation matrix can be obtained as a function of the nodal coordinates (Brenner 1995). The random field is conditioned by assuming the support conditions as deterministic. The modified *conditional* random field (Vanmarcke 1983, Ditlevsen 1991) is no longer weakly homogeneous. Its parameters are determined via a stochastic interpolation scheme which is based on the maximum likelihood principle (Ditlevsen 1991).

The final correlation matrix $\hat{\mathbf{C}}_{xx}$ is diagonalized:

$$\mathbf{\Psi}^T \hat{\mathbf{C}}_{xx} \mathbf{\Psi} = diag\left(\sigma_{Y_i}^2\right) \text{ with } \sigma_{Y_1}^2 \geq \sigma_{Y_2}^2 \geq ...\sigma_{Y_n}^2 \quad (1)$$

The eigenvectors $\mathbf{\Psi}$ characterize the imperfection shapes, the eigenvalues $\sigma_{Y_i}^2$ represent the variances of the respective amplitudes. These amplitudes are normally distributed, have zero mean and are ordered with decreasing size (Brenner 1995).

The failure probability of the structure is computed by integration of the marginal distribution of the random variable vector $\mathbf{Y}$ over the failure domain indicated by $g(\mathbf{y}) \leq 0$:

$$p_f = \int_{g(\mathbf{y})<0} f_{\mathbf{Y}}(\mathbf{y}) d\mathbf{y}, \quad (2)$$

where $g(\mathbf{y}) \leq 0$ indicates the region of instability. To solve Eq.2 imperfection shapes are increased until the stability border is reached. When $f_{\mathbf{Y}}$ is of dimension one the failure probability can be obtained analytically. To investigate a higher dimensional problem an interaction model can be applied. The stability boundaries can be computed with the linear and the nonlinear method.

2.1.2 Random Excitation

The random excitation process is assumed to be a *white noise* process with a given mean value and a power spectral density. The process is dicretized by using Fourier series (FFT) with Fourier coefficients as zero-mean Gaussian random variables and random amplitudes according to Rice 1948.

2.2 Mechanical Model

2.2.1 Reference Solution and Consistent Linearization

The nonlinear equation of motion of a system can be written in a matrix-vector equation:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{r}(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{f} \quad (3)$$

with the mass matrix $\mathbf{M}$, the the nonlinear restoring force vector $\mathbf{r}$, depending on the nodal displacement vector $\mathbf{x}$, and the time depending continuous loading function $\mathbf{f}$. Eq.3 is valid for any perfect or imperfect structural system. The formal linearization of the nonlinear restoring forces, which are supposed to be continuous and differentiable, with respect to a continuous reference solution $\mathbf{x}_0$ and $\dot{\mathbf{x}}_0$ yields:

$$\mathbf{r} = \mathbf{r}\left(\mathbf{x}_0, \dot{\mathbf{x}}_0\right) + \left.\frac{\partial \mathbf{r}}{\partial \dot{\mathbf{x}}}\right|_{\mathbf{x}_0, \dot{\mathbf{x}}_0} \dot{\mathbf{y}} + \left.\frac{\partial \mathbf{r}}{\partial \mathbf{x}}\right|_{\mathbf{x}_0, \dot{\mathbf{x}}_0} \mathbf{y}, \quad (4)$$

or

$$\mathbf{r} = \mathbf{r}\left(\mathbf{x}_0, \dot{\mathbf{x}}_0\right) + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} \quad (5)$$

with the deviation from the reference solution $\mathbf{y} = \mathbf{x} - \mathbf{x_0}$ and the tangential stiffness and the damping matrices $\mathbf{K}$ and $\mathbf{C}$ respectively. The equation of motion can be split into a differential equation for the reference solution itself,

$$\mathbf{M}\ddot{\mathbf{x}}_0 + \mathbf{r}(\mathbf{x}_0, \dot{\mathbf{x}}_0) = \mathbf{f} \quad (6)$$

and a differential equation for the difference to neighboring motions:

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{0} \quad (7)$$

2.2.2 Nonlinear Stability Analysis

To analyze the dynamic stability behaviour of nonlinear systems an integration of Eq.6 is necessary until stochastic stationarity is reached. In each time step, the tangential stiffness matrix $\mathbf{K}$ has to be determined. With this kind of analysis a criterion for sample stability is developed. In order to speed up explicit time integration, this equation can be projected into a subspace of dimension m as spanned by the eigenvectors of the undamped system corresponding to the m smallest natural frequencies (Bucher 2001). These eigenvectors are the solutions to

$$\left(\mathbf{K}\left(\mathbf{x}_{stat}\right) - \omega_i^2 \mathbf{M}\right) \mathbf{\Phi} = 0; \quad i = 1 \ldots m \quad (8)$$

In this equation, $\mathbf{x}_{stat}$ is chosen to be the displacement solution of Eq.6 under static loading conditions. The mode shapes are assumed to be mass normalized. A transformation $\mathbf{x} = \mathbf{\Phi}\mathbf{v}$ and a multiplication of Eq.6 with $\mathbf{\Phi}^T$ represents a projection of the differential equation of motion for the reference solution into the subspace of dimension m as spanned by the eigenvectors:

$$\ddot{\mathbf{v}} + \mathbf{\Phi}^T \mathbf{r}(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{\Phi}^T \mathbf{f} \quad (9)$$

The integration of Eq.9 by the central difference method (Bathe 1996) requires a minimal time step.

The time integration in the subspace and the computing of the restoring forces on the full system causes the following problem: If the initial displacement or velocity vector of the time integration is not zero, for example due to static loading, the projection of these vectors into the subspace is an optimiziation problem caused by the higher number of variables in the full space. A possibility to improve the situation, is to start the time integration in the subspace with a displacement and velocity vector equal to zero. The initial vectors have to be saved in the full system and the restoring force vector has to be computed by addition of the initial and the time integration vectors:

$$\mathbf{r}(\mathbf{x},\dot{\mathbf{x}}) = \mathbf{r}(\mathbf{x}_{start} + \mathbf{\Phi}\mathbf{v}, \dot{\mathbf{x}}_{start} + \mathbf{\Phi}\dot{\mathbf{v}}); \quad \mathbf{v}(t=0) = \dot{\mathbf{v}}(t=0) = \mathbf{0} \tag{10}$$

In the investigated cases the initial vector $\mathbf{x}_{start}$ is the static displacement vector, the initial velocities are assumed to be zero.

To analyze the stability behaviour of the reference solution $\mathbf{x}_0(t)$, the long-term behavior of the neighboring motion (Eq.7) is investigated. To reduce the dimension of the equation system, this equation can be projected into the same or a smaller subspace as Eq.9. Transformed into the state space description we obtain:

$$\dot{\mathbf{z}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi} & -\mathbf{\Phi}^T\mathbf{C}\mathbf{\Phi} \end{bmatrix} \mathbf{z} = \mathbf{A}[\mathbf{x}_0(t)]\mathbf{z} \tag{11}$$

From this equation, the Lyapunov exponent λ can be determined by a limiting process:

$$\lambda(\mathbf{x}_0,\mathbf{s}) = \lim_{t\to\infty} \frac{1}{t} \log ||\mathbf{\Theta}(\mathbf{x}_0,t)\mathbf{s}|| \tag{12}$$

in which $\mathbf{s}$ is an arbitrary unit vector. In Eq.12, $\mathbf{\Theta}(\mathbf{x}_0,t)$ is the transition matrix from time 0 to t associated with Eq.11. Based on the multiplicative ergodic theorem (e.g. Arnold and Imkeller 1994) the Lyapunov exponent can also be calculated as an expected value:

$$\lambda(\mathbf{x}_0,\mathbf{s}) = E[\frac{d}{dt} \log ||\mathbf{\Theta}(\mathbf{x}_0,t)\mathbf{s}||] \tag{13}$$

In the current investigation, the norm $||\mathbf{\Theta}(\mathbf{x}_0,t)\mathbf{s}||$ is expressed in terms of

$$\|\mathbf{\Theta}(\mathbf{x}_0,t)\mathbf{s}\| \le \|\mathbf{\Theta}(\mathbf{x}_0,t)\| \cdot \|\mathbf{s}\| = \|\mathbf{\Theta}(\mathbf{x}_0,t)\| \tag{14}$$

Finally, this result is used in calculating the Lyapunov exponent according to Eq.12 by using a matrix norm equal to the eigenvalue μ_{max} of $\mathbf{\Theta}(\mathbf{x}_0,t)$ with the maximum absolute value. The time domain t has to be taken large enough that the Lyapunov exponent convergences to a stationary value. For the statistical estimation of the convergence of the Lyapunov exponent, Eq.13 is suitable.

2.2.3 Linear Stability Analysis

The Lyapunov exponent for the stability of the second moments of a linearized reference solution can be determined by the Itô analysis. The nonlinear stiffness matrix in Eq.7 can be expanded into an asymptotic series with respect to a static loading condition. Under the assumption that the fluctuating part is small enough this series can be truncated after the linear term:

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + (\mathbf{K}(\mathbf{x}_{stat}) + \mathbf{f}(t)\mathbf{K}_1)\mathbf{y} = \mathbf{0} \tag{15}$$

This equation of motion is projected into a subspace of dimension m and then transformed into its state space description analogous to Eq.11:

$$\dot{\mathbf{z}} = [\mathbf{A} + \mathbf{B}\mathbf{f}(t)]\,\mathbf{z} \tag{16}$$

where the coefficient matrices $\mathbf{A}$ and $\mathbf{B}$ are constant. $\mathbf{f}(t)$ is assumed to be Gaussian white noise. Then the Eq.16 represents a first order stochastic differential equation. For this system the Lyapunov exponent λ_2 for the second moments can be easily derived by applying the Itô calculus (e.g. Soong and Grigoriu 1992, Lin and Cai 1995), which leads to a linear ordinary differential equation for the covariance expressions $\mathbf{C}_{\mathbf{zz}} = E\left[\mathbf{z}\mathbf{z}^T\right]$ depending on the noise intensity D_0:

$$\dot{\mathbf{C}}_{\mathbf{zz}} = \mathbf{F}\mathbf{C}_{\mathbf{zz}} + \mathbf{C}_{\mathbf{zz}}\mathbf{F}^T + D_0\mathbf{B}\mathbf{C}_{\mathbf{zz}}\mathbf{B}^T, \quad \mathbf{F} = \mathbf{A} + \frac{D_0}{2}\mathbf{B}^2 \tag{17}$$

By rearranging $\mathbf{C}_{\mathbf{zz}}$ in a vector, the equation can be rewritten in matrix-vector-form, where the largest real part of the eigenvalues of this matrix represents λ_2.

The Lyapunov exponents for almost sure stability can be approximated for linear SDOF-systems analytically (Lin and Cai 1995):

$$\lambda = -\zeta_0\omega_0 + \frac{\pi S_{ff}\omega_0^2}{4} \tag{18}$$

where ω_0 is the natural frequency, ζ_0 is the modal damping ratio and S_{ff} is the power spectral density of the white noise excitation. By exploiting the equation

$$\lambda_2 = -2\zeta_0\omega_0 + \pi S_{ff}\omega_0^2 \tag{19}$$

the Lyapunov exponent for the samples can be approximated from λ_2 according to

$$\lambda = \frac{\lambda_2}{4} - \frac{\zeta_0\omega_0}{2} \tag{20}$$

This equation can also be applied on MDOF-systems, it should be mentioned that the term $-\zeta_0\omega_0$ corresponds then to solution without random parametric excitation.

3 NUMERICAL EXAMPLES

3.1 A nonlinear simple column

The nonlinear behaviour of a imperfect structure was investigated on a simply supported column subjected to a random vertical load as shown in Fig.1.

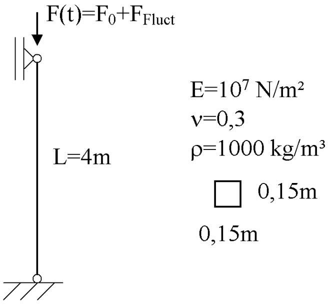

Figure 1: Investigated MDOF-Column

The static load F_0 is chosen to be 80% of the critical load of the perfect column. A nonlinear static stability analysis leads to the value of $F_{crit} = 259.2N$. The dynamic load is chosen to be a Gaussian white noise. For the dynamic stability analysis, a modal damping ratio of $\zeta_k = 0.01$ is assumed for all modes.

To reduce the numerical effort a modal reduction is necessary. This requires an investigation which dimension of the modal subspace is necessary to reproduce the stability behaviour. Be reducing the full system with $m = 60$ modes to $m = 20$, where a sufficient comparisson is granted, the critical time steps were one order of magnitude apart so that $m = 20$ led to a speed-up of 10.

Investigations of the reduced perfect system have shown, that the assumption of the linear method of similar time behaviour of the stiffness matrix and the excitation is not exact, caused by a quite substantial difference in the time fluctuations of stiffness matrix and excitation. However, for a first approximation of the stability boundary the linear analyses can be applied. The imperfection influence was investigated on a imperfection shape proportional to the first buckling shape for a fixed load factor $\ell = 200N$. The magnitudes of mid-span imperfections are varied from 0 to $2.79\ cm$. The destabilizing influence of the geometrical imperfections is easily seen from the results as given in Fig.2. The figure shows that the Lyapunov exponents obtained by the linear and nonlinear analysis are in the same dimension in the stable area. In the unstable region the difference is quite substantial.

3.2 Reliability investigation of a shell structure

A cylindrical panel was considered, which is mentioned e.g. in Krätzig 1989 and Schorling and Bucher 1999. The assumed structure is shown in Fig.3. The geometrical and the material proberties were given as: radius $R = 83.33m$, the half width and heigth $a = 5m$, the thickness $h = 0.1m$, the Young's modulus $E =$

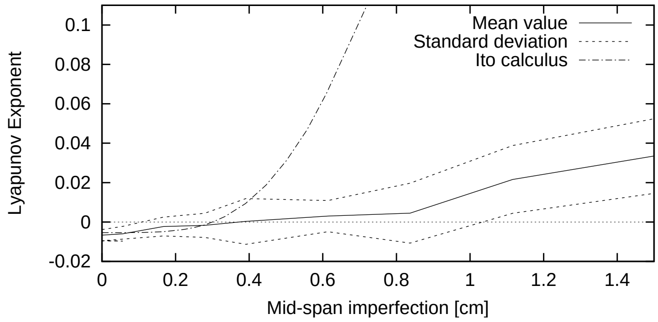

Figure 2: Comparison of nonlinear and Itô-analysis

$3.4 10^{10} N/m^2$, the mass density $\rho = 3400kg/m^3$ and the Poisson's ratio $\mu = 0.2$. The constant load factor is $P = 1000N/m$.

The structure is discretized with 7×7 nodes and meshed with geometrically nonlinear 9-node shell elements. At a static load factor of $\nu_{crit} = 16825$ the structure reaches an unstable state (Krätzig 1989: $\nu_{crit} = 15120$, Schorling and Bucher 1999: $\nu_{crit} = 16200$; both used a different discretisation and different elements). The static load is assumed to be $P_0 = 0.85\nu_{crit}P$. The fluctuating load is considered as $P_{fluct} = \ell f(t)P$, where $f(t)$ is the unit white noise process and ℓ is the load factor. The damping is assumed as modal damping with the damping ratio $\zeta_k = 0.02$ for all modes.

The geometrical imperfections are considered in terms of radial deviations from the perfect panel surface and are modelled as a conditional Gaussian random field. The mean is assumed as zero and the standard deviation as $\sigma = 10^{-3}m$. The correlation length of the exponential correlation function is considered with $l_H = 10m$. The imperfection shapes are obtained by the decomposition of the covariance matrix according to Eq.1. The first four imperfection shapes are shown in Fig.3 as well. The corresponding standard deviations σ_{Y_i} in uncorrelated normal space are indicated in the figure. The first shape is very similar to the buckling shape.

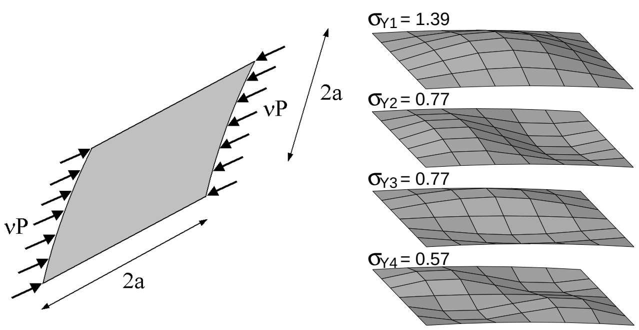

Figure 3: Nonlinear cylindrical shell structure with associated weighted imperfection shapes

The structure was investigated by using the Itô analysis and it was found that only the first imperfection shape has a major influence on the stability behaviour. The critical noise intensity of the perfect system was obtained as $D_{0_{crit}} = 92000\pi$ with the linear and $D_{0_{crit}} = 20000\pi$ with the nonlinear method by averaging 20 simulations with 10^5 time steps. The nonlinear analysis uses a modal subspace spanned by 12 of the 213 eigenmodes with a critical time step of $\Delta t = 6.3 \cdot 10^{-3}s$. The investigation of the first imperfection shape obtained by nonlinear analysis show observable deviations from the linear results. This points out that the nonlinearities of this structure have a higher influence as compared to the previous example. The obtained stability boundaries depending on the imperfection size are displayed in Fig.4 for the linear and the nonlinear analysis.

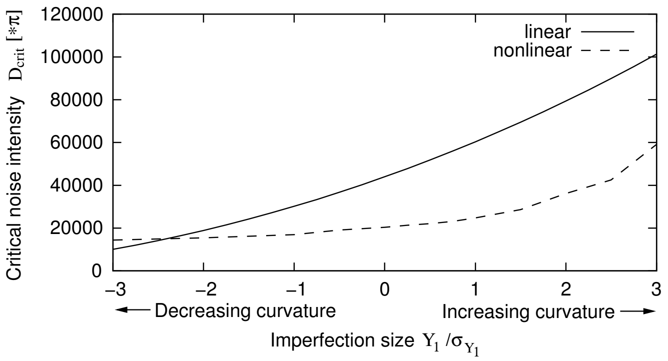

Figure 4: Stability boundaries vs. imperfection size

The failure probability for this one dimensional problem can be obtained analytical from the stability boundaries and is shown in Fig.5 depending on the noise intensity for both methods. It is to be seen in the picture, that a sufficient approximation of the nonlinear probability graph is not possible with the linear method.

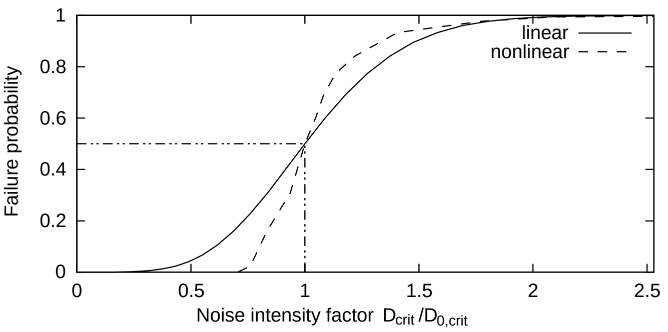

Figure 5: Failure probabilities for both methods

Furthermore the discretisation influence on the stability boundaries was investigated on the perfect panel. Additional to the 7×7 node model, systems modeled with 11×13 (Schorling and Bucher 1999) and 25×25 nodes and meshed with geometrically nonlinear 9-node shell elements were analyzed. The critical static buckling loads are shown in Table 1. The dynamic stability boundaries are obtained by using the linear Itô analysis under considering the static load first with $0.85\nu_{crit}$ of the same model. This leads to different static loads. To obtain the stability boundaries by a constant static load this load was assumed as $0.85\nu_{crit}$ of the 7×7 node model. The results are shown additional in Table 1. It is to be seen that the influence of the discretization on the critical noise intensity is much higher than on the static buckling load. The nonlinear method was not applicable for the 11×13 and 25×25 node models, caused by the hugh numerical effort.

Model	ν_{crit}	$D_{0_{crit}}$ $\nu = 0.85\nu_{crit}$	$D_{0_{crit}}$ $\nu = 0.85\nu_{crit,7\times7}$
7×7	16825	91736π	91736π
11×13	15968	43946π	32116π
25×25	15744	32353π	13854π

Table 1: Critical static and dynamic loads

4 CONCLUSIONS

The paper presents two methods to analyze the stochastic dynamic stability behaviour of structure which are discretized by finite element models. The nonlinear method can consider geometrical and material nonlinearities by using a nonlinear explicit time integration. The linear method is only applicable for linear or linearized systems. In the presented examples it was shown that the influence of nonlinearities could be very varying for different types of structures. The linear method does not necessarily give approximately correct stability boundaries.

For reliability analysis the random imperfections are represented by a conditional random field, whose correlation matrix can be diagonalized in different imperfection shapes and respective random amplitudes. The failure probability can be computed depending on the dimension of the random variable vector.

It is necessary to compute many stability boundaries from different imperfection-size/excitation-intensity combinations for every imperfection shape. With the nonlinear method this number of necessary simulations is not realizable for larger systems caused by the numerical effort. It is suggested to use the linear method to find the imperfections shapes which generally influence the stability behaviour. The stability boundaries can be approximated with the linear method as well, but the nonlinear method should by applied then to check or if necessary to correct the results. The presented examples indicate the importance of appropriate discretization to reproduce the dynamic stability behaviour of the system, correctly.

5 ACKNOWLEDGEMENT

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