

# BAYESIAN RELIABILITY UPDATING USING SYSTEM IDENTIFICATION BASED ON SELECTIVE SENSITIVITY

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## Abstract

A new selectively sensitive approach for system identification is presented. Together with a Bayesian updating method this leads to an improved reduction of system uncertainty. A numerical example demonstrates that the improvement is significant.

## Introduction

The assessment of structural reliability in the context of health monitoring or design re-evaluation must be based on a suitable probabilistic description of structural properties. During the life-span of a structure this amounts to observing relatively small changes of system properties. Due to their simplicity, measurements of dynamic responses are most suitable for permanent observations. These measurements have to be followed by the (statistical) estimation of parameter values for a given structural model (Macke and Bucher 2002). Real structures generally exhibit random spatial fluctuations of their properties which are best described by random fields. As an example, concrete degradation has recently been modeled in terms of random fields (Li et al. 2004). It is well known, that the number of random variables required to represent a random field increases dramatically with decreasing correlation length of the field, e.g. (Matthies and Bucher 1999). Consequently, an appropriate updating procedure may require a large number of parameter values to be identified simultaneously. It has been shown by (Bucher et al. 2003), that failure to consider spatial variability in system identification may lead to gross over-estimation of structural safety. Unfortunately, system identification requires the solution of inverse problems, which usually leads to rather ill-conditioned mathematical formulations, e.g. linear equations. The consequence of the ill-conditionedness is that any small errors in the measurements will be considerably amplified, so that finally large errors in the identified parameter values will occur. The concept of Selective Sensitivity (Prells and Ben-Haim 1993), (Ben-Haim 1996) theoretically allows a way out of this dilemma. Basically, this approach provides specific excitations which are quite sensitive to small number of parameters (which are identified) and rather insensitive to a large number of parameters (which are not identified). By suitable repetition of the procedure, all parameter values can be found. The present paper suggests an alternative approach for static identification which is appropriate for statically determinate structures. The advantage of this new approach is that no prior knowledge of the system parameters is required to obtain selectively sensitive loading patterns. This new approach is then combined with Bayesian updating to provide significantly reduced uncertainty of the system parameters.

## Selective Sensitivity

The equation of motion of an undamped MDOF-system is given in the following form:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{f}(t) \quad (1)$$

in which  $\mathbf{M}$  denotes the mass matrix,  $\mathbf{K}$  is the stiffness matrix,  $\mathbf{f}$  is the external excitation, and  $\mathbf{x}$  is the displacement response. Generally it is assumed that the stiffness matrix can be written in the form

$$\mathbf{K} = \sum_{m=1}^M a_m \mathbf{K}_m \quad (2)$$

in which the coefficients  $a_m$  have to be determined from the identification procedure. The purpose of selective sensitivity is to provide excitation vectors  $\mathbf{F}_p$  in such a way, that the sensitivity of measured responses to changes in the parameters  $a_m$  becomes (almost) zero for all  $m \neq p$ . However, this step generally involves the knowledge of all parameters to be identified, thus possibly losing the advantages of the selectivity.

## A New Approach

Consider the statically determinate simply supported beam with span length  $L$  as shown in Fig. 1. We want to determine the bending stiffness values  $EI_m$ ;  $m = 1 \dots 4$  using selectively sensitive load configurations  $F_l$ ;  $l = 1 \dots 4$ . Setting all loads to the same

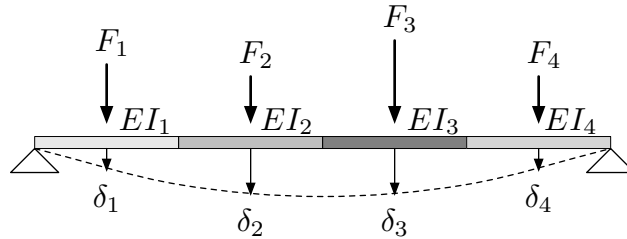


Figure 1: Simply supported beam

value  $F_0$ , the relationship between displacements  $\boldsymbol{\delta} = [\delta_1, \dots, \delta_4]^T$  and element bending stiffnesses  $\mathbf{h} = [\frac{1}{EI_1}, \dots, \frac{1}{EI_4}]^T$  is given by  $\boldsymbol{\delta} = \mathbf{H}_0 \mathbf{h}$  in which the matrix  $\mathbf{H}_0$  is defined by

$$\mathbf{H}_0 = \begin{bmatrix} 0.005086 & 0.009073 & 0.005574 & 0.001099 \\ 0.005493 & 0.023965 & 0.016722 & 0.003296 \\ 0.003296 & 0.016723 & 0.023964 & 0.005493 \\ 0.001099 & 0.005574 & 0.009073 & 0.005086 \end{bmatrix} \cdot F_0 L^3 \quad (3)$$

This matrix has a condition number of 33, which already quite clearly indicates the problems that generally arise from the simultaneous identification of many parameters. In this particular case, if we assume the displacements to be measurable with an accuracy (coefficient of variation) of 2%, then a Monte Carlo study shows that the identified stiffness values have coefficients of variation of 20% ( $EI_1$  and  $EI_4$ ) and 14% ( $EI_2$  and  $EI_3$ ). So the

uncertainty of the measurements is propagated and magnified in an extremely bad way rendering the Bayesian updating process rather ineffective. Due to the fact that the structure is statically determinate, i.e. the distribution of the bending moments is independent of the actual bending stiffness values, it is possible to write the flexibility matrix  $\mathbf{H}$  relating the forces  $F_l$  to the displacements  $\delta_k$  by means of  $\delta = \mathbf{H}\mathbf{F}$  as follows:

$$\mathbf{H} = \sum_{m=1}^4 \mathbf{H}_m \frac{1}{EI_m} \quad (4)$$

so that

$$\delta_k = \sum_{l,m=1}^4 H_{klm} F_l \frac{1}{EI_m}; k = 1 \dots 4 \quad (5)$$

where the flexibility contributions of each element  $\mathbf{H}_m$  are given by

$$\mathbf{H}_1 = \begin{bmatrix} 0.001790 & 0.001831 & 0.001099 & 0.000366 \\ 0.001831 & 0.002034 & 0.001221 & 0.000407 \\ 0.001099 & 0.001221 & 0.000732 & 0.000244 \\ 0.000366 & 0.000407 & 0.000244 & 0.000081 \end{bmatrix} \cdot L^3 \quad (6)$$

$$\mathbf{H}_2 = \begin{bmatrix} 0.001546 & 0.003947 & 0.002685 & 0.000895 \\ 0.003947 & 0.010416 & 0.007202 & 0.002401 \\ 0.002685 & 0.007202 & 0.005127 & 0.001709 \\ 0.000895 & 0.002401 & 0.001709 & 0.000570 \end{bmatrix} \cdot L^3 \quad (7)$$

Note that the remaining matrices  $\mathbf{H}_3$  and  $\mathbf{H}_4$  are easily obtained from symmetry considerations. It is now possible to choose the elements of a force vector  $\mathbf{F}^{(k)}$  in such a way, that one specific displacement  $\delta_k$  depends only on the stiffness  $EI_k$  but not on any other stiffness. This can be achieved by assembling the  $k$ -th rows of the matrices  $\mathbf{H}_m$  into a matrix  $\mathbf{B}_k$  and solve the system of equations

$$\mathbf{B}_k \mathbf{F}^{(k)} = \mathbf{u}_k \quad (8)$$

in which  $\mathbf{u}_k$  is a vector with only one arbitrary, non-zero entry at the  $k$ -th position. For  $k = 1$ , this leads to a force vector  $\mathbf{F}^{(1)}$ :

$$\mathbf{F}^{(1)} = [ 1.00000 \quad -0.44751 \quad 0.08960 \quad -0.02260 ]^T \cdot F_0 \quad (9)$$

Here the  $k$ -th element of  $\mathbf{F}^{(1)}$  has been set to a reference value of  $F_0$ , since the scaling is arbitrary. Applying this load vector to the structure, it is seen that the displacement  $\delta_1$  becomes

$$\delta_1 = 0.0010609 F_0 L^3 / EI_1 \quad (10)$$

from which the bending stiffness  $EI_1$  is readily computed. In the same manner, force vectors and displacement relations involving the remaining bending stiffnesses can be obtained as

$$\mathbf{F}^{(2)} = [ -0.75079 \quad 1.00000 \quad -0.59320 \quad 0.15969 ]^T \cdot F_0; \quad \delta_2 = 0.0035631 F_0 L^3 / EI_2 \quad (11)$$

The remaining equations for  $EI_3$  and  $EI_4$  are easily obtained by symmetry considerations. These relations clearly show that now there is no magnification of the measurement error.

## Bayesian Updating

The Bayesian probabilistic framework for model updating has been presented in the work by (Beck and Katafygiotis 1998). In this paper, the Bayesian updating procedure is applied incorporating with the sensitively selective static forces which have been determined in the previous section. For convenience, the bending stiffnesses are parameterized by  $\frac{1}{EI_m} = a_m \times \frac{1}{EI_{0,m}}$ ;  $m = 1 \dots 4$ , where each  $EI_{0,m}$  is a nominal bending stiffness,  $\mathbf{a} = [a_1, \dots, a_4]^T$  are the model parameters which will be updated. Given the prior probability  $p'(\mathbf{a})$ , the updated (posterior) probability  $p''(\mathbf{a})$  when gaining some measured data  $\mathbf{y}$  is obtained by

$$p''(\mathbf{a}) = c \times p'(\mathbf{a})p(\mathbf{y}|\mathbf{a}) \quad (12)$$

where  $p(\mathbf{y}|\mathbf{a})$  is the PDF of the data given the model parameters  $\mathbf{a}$ , and  $c$  is a normalizing constant. Assume that the structural model,  $\boldsymbol{\delta}(\mathbf{a}) = \mathbf{H}(\mathbf{a})\mathbf{F}$ , is exact, however the measured value  $\mathbf{y}$  of the displacements is in error. Denote  $e(n)$  the error in the  $n$ -th ( $n = 1 \dots N_s$ ) observation, then we have

$$\mathbf{y}(n) = \mathbf{H}(\mathbf{a})\mathbf{F} + e(n) \quad (13)$$

The random error term  $e(n)$  is assumed normally distributed with zero-mean and covariance matrix  $C(n)$ . If the measurements are independent with COV of  $\psi$ , the covariance matrix becomes a diagonal matrix with all diagonal elements equal to  $\psi^2(\mathbf{y}^T(n)\mathbf{y}(n))$ . Therefore, the resulting PDF for  $\mathbf{y}$  is

$$p(\mathbf{y}|\mathbf{a}) = c_1 \exp \left[ -\frac{1}{2} \sum_{n=1}^{N_s} \frac{(\mathbf{y}(n) - \boldsymbol{\delta}(\mathbf{a}))^T (\mathbf{y}(n) - \boldsymbol{\delta}(\mathbf{a}))}{\psi^2 \mathbf{y}^T(n)\mathbf{y}(n)} \right] \quad (14)$$

Let us assume that the prior density functions of the parameters  $a_m$  are Gaussian with mean  $\mu_m$ ; standard deviation  $\sigma_m$ , and they are mutually independent. Thus, the prior joint PDF of the parameter vector  $\mathbf{a}$  is

$$p'(\mathbf{a}) = c_2 \exp \left[ -\frac{1}{2} \sum_{m=1}^4 \frac{(a_m - \mu_m)^2}{\sigma_m^2} \right] \quad (15)$$

resulting in the updated PDF of the parameter  $\mathbf{a}$

$$p''(\mathbf{a}) = c \times \exp [-J(\mathbf{a})] \quad (16)$$

where the overall measure of fit (MOF),  $J(\mathbf{a})$ , is

$$J(\mathbf{a}) = \frac{1}{2} \sum_{m=1}^4 \frac{(a_m - \mu_m)^2}{\sigma_m^2} + \frac{1}{2} \sum_{n=1}^{N_s} \frac{(\mathbf{y}(n) - \boldsymbol{\delta}(\mathbf{a}))^T (\mathbf{y}(n) - \boldsymbol{\delta}(\mathbf{a}))}{\psi^2 \mathbf{y}^T(n)\mathbf{y}(n)} \quad (17)$$

The posterior PDF of each parameter  $a_m$  can be approximated using Laplace's method for asymptotic expansion (Papadimitriou et al. 1997)

$$p''(a_m) \approx \frac{1}{\sigma_m^* \sqrt{2\pi}} \exp \left[ -\frac{(a_m - a_m^*)^2}{2(\sigma_m^*)^2} \right] \quad (18)$$

where  $\mathbf{a}^*$  is the most probable model obtained by minimizing  $J(\mathbf{a})$  in Eq. 17, and the variance  $(\sigma_m^*)^2$  is the  $m$ -th diagonal element of  $\mathbf{L}(\mathbf{a}^*)^{-1}$  with  $\mathbf{L}(\mathbf{a}^*)$  is the Hessian matrix of  $J(\mathbf{a})$  calculated at  $\mathbf{a}^*$ . As stated above, by applying the selectively sensitive force vector  $\mathbf{F}^{(k)}$ , the bending stiffness  $EI_k$ , however, can be separately determined using only the single measured displacement value  $y_k$ . Thus, the updated PDF of the parameter  $a_k$  is

$$p''(a_k) = c_k \times \exp \left[ -\frac{1}{2} \frac{(a_k - \mu_k)^2}{\sigma_k^2} - \frac{1}{2} \sum_{n=1}^{N_s} \frac{(y_k(n) - \delta_k)^2}{\psi^2 y_k^2(n)} \right] \quad (19)$$

where

$$\delta_k = \sum_{l=1}^4 H_{kl} F_l^{(k)} = \delta_k(a_k) \quad (20)$$

Given are the nominal bending stiffness  $EI_{0,1}, EI_{0,2}, EI_{0,3}, EI_{0,4} = 1.0Nm^2$ , the initial model parameter  $\mathbf{a}_0 = [\mu_1, \mu_2, \mu_3, \mu_4]^T = [1, 1, 1, 1]^T$  to reflect that the nominal structural model is the most probable model in the absence of any data (Vanik et al. 2000). The beam has the length  $L = 1.0m$ ; and the actual bending stiffness  $EI_1 = 0.8Nm^2; EI_2, EI_3 = 0.9Nm^2; EI_4 = 0.7Nm^2$ , thus the actual values of the parameters  $\mathbf{a} = [1.25, 1.11, 1.11, 1.43]^T$ . The standard deviations of the prior density functions  $\sigma_m$  equal to 0.1 for all parameters. Noisy measurement with 2% coefficient of variation is assumed, i.e  $\psi = 0.02$ . The force  $\mathbf{F}$  is set to a reference value  $F_0 = 1.0N$ . The parameters are updated first simultaneously by applying a force vector  $\mathbf{F} = 0.1F_0[1, 1, 1, 1]^T$ , and then separately by using the selectively sensitive force (sel-force) vectors. The updated mean values and COVs of the parameters  $a_1$  and  $a_2$  taking 1, 5 and 10 observations are listed in Table 1. It can be seen that the COVs obtained by using sel-force are much smaller than those obtained without sel-force. As more observations are taken, the COVs decrease. In addition, the mean values are also effectively updated with sel-force. Fig. 2a and 2b also show the prior as well as posterior PDFs of the parameters  $a_1$  and  $a_2$ , respectively. It is shown that using the selectively sensitive forces efficient reduction of the uncertainty can be obtained.

Table 1: Updated mean values and COVs of  $a_1$  and  $a_2$  with  $N_s$  observations

Number of Observation	Parameter	w/o sel-force		w/ sel-force	
		mean	COV	mean	COV
1	$a_1$	1.05275	9.21%	1.24556	1.95%
	$a_2$	1.20803	5.94%	1.08922	1.92%
5	$a_1$	1.06187	8.58%	1.25266	1.06%
	$a_2$	1.16032	4.78%	1.11261	0.89%
10	$a_1$	1.07714	8.02%	1.24813	0.77%
	$a_2$	1.14981	4.14%	1.10641	0.64%

## Conclusions

It has been shown that for statically determinate beam structures it is easily possible to determine selectively sensitive loading patterns for static system identification. This approach allows to significantly reduce the propagation and amplification of measurement

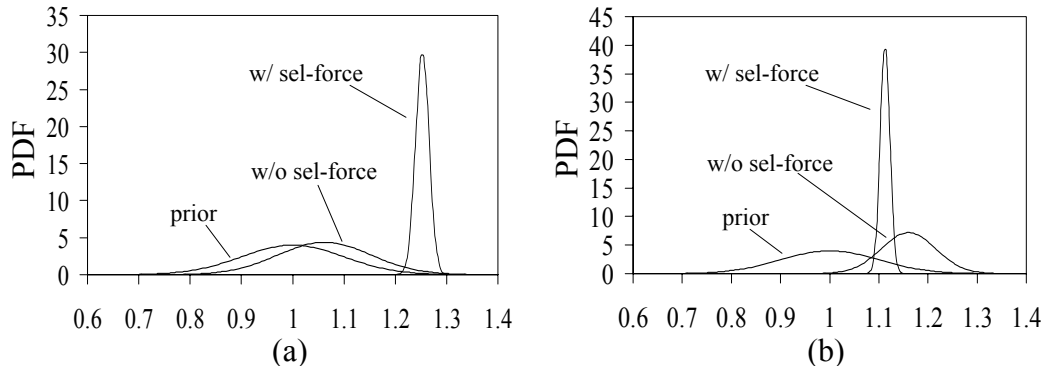


Figure 2: Posterior PDFs of : (a) parameter  $a_1$  and (b) parameter  $a_2$  with  $N_s = 5$

errors to the identified parameters. In this way, the Bayesian updating procedure becomes more effective and reliable. Future work will focus on the extension to dynamic testing by including inertia forces into the analysis.

### Acknowledgements

This work has been carried out within the Collaborative Research Center SFB 524 supported by the German Research Council (DFG). The second author has been supported by a scholarship of the German Academic Exchange Service (DAAD). All support is gratefully acknowledged.

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